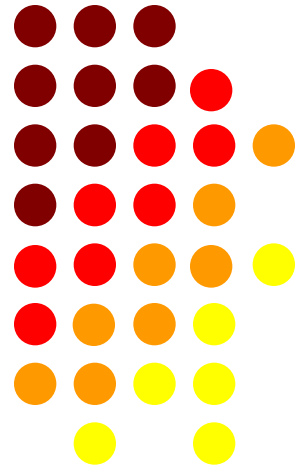


Unit IV
(Multivariate Calculus-I)
L-27

**Introduction to
double integrals**



Introduction :- In this unit we will study about double & triple integrals, which are very useful in finding area, volume, mass, centroid etc.



Double Integral $\Rightarrow$ An integral of the form $I = \iint_R f(x,y) dx dy$ is called double integral of $f(x,y)$ over the region R , which can also be written as

$$I = \iint_R f(x,y) dy dx.$$

Evaluation of Double Integral $\Rightarrow$ The method of evaluating the double integrals depend upon the nature of the curves bounding the region R .



case-a) $\Rightarrow$ If the region R is defined as $\Rightarrow$

$$a \leq x \leq b, \quad f_1(x) \leq y \leq f_2(x)$$

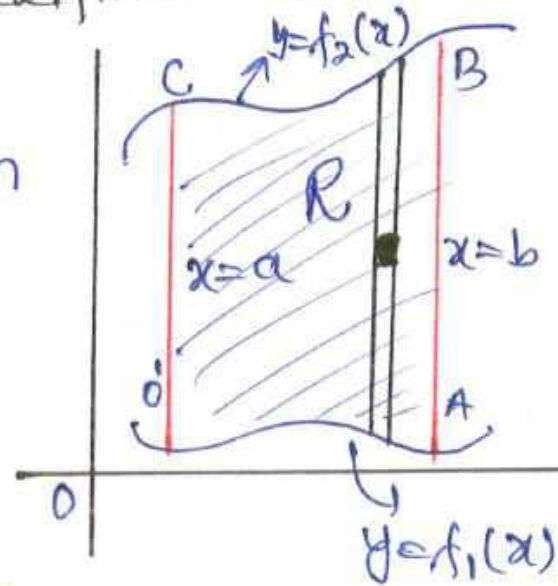
Here region R is the region represented by $O'ABC$,

first we will take a vertical strip & treating x as a constant. Then we

have lower limit of x is a
lower limit of y is $f_1(x)$

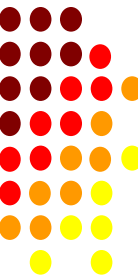
upper limit of x is b

& upper limit of y is $f_2(x)$.



Now the double integral is evaluated first w.r.t y (treating x as a constant) & then w.r.t x , then we have

$$\iint_R f(x,y) dx dy = \int_{x=a}^b \left[\int_{y=f_1(x)}^{f_2(x)} f(x,y) dy \right] dx$$



Ques → Evaluate $\int_0^a \int_0^x xy \, dy \, dx$ [2013-14]

Solⁿ → here $[0=f_1(x)] \leq y [x=f_2(x)]$
 $\& [0 \leq x \leq a]$

$$\begin{aligned}
 \text{so } I &= \int_{x=0}^a \int_{y=0}^x xy \, dy \, dx \\
 &= \int_{x=0}^a x \left[\int_{y=0}^x y \, dy \right] dx = \int_{x=0}^a x \left(\frac{y^2}{2} \right)_0^x dx \\
 &= \int_{x=0}^a x \cdot \left[\frac{x^2}{2} - 0 \right] dx = \frac{1}{2} \int_{x=0}^a x^3 \, dx \\
 &= \frac{1}{2} \left(\frac{x^4}{4} \right)_0^a = \frac{a^4}{8} \quad \text{Ans.}
 \end{aligned}$$



Ques: Evaluate $\int_0^a \int_{x/a}^x \frac{x}{x^2 + y^2} dy dx$

Sol: $I = \int_0^a x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_{x/a}^x dx$ $\left[\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$

$$= \int_0^a x \left[\frac{1}{x} \tan^{-1} 1 - \frac{1}{x} \tan^{-1} \frac{1}{a} \right] dx = \int_0^a \left(\frac{\pi}{4} - \tan^{-1} \frac{1}{a} \right) dx$$

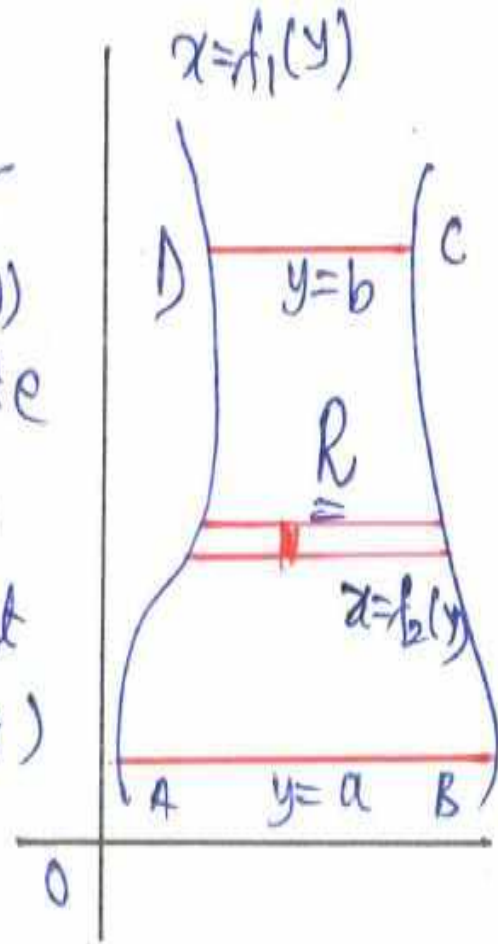
$$= \left(\frac{\pi}{4} x - x \tan^{-1} \frac{1}{a} \right)_0^a = \frac{\pi}{4} a - a \tan^{-1} \frac{1}{a} \quad \text{Ans}$$



Case: b) $\Rightarrow$ If Region R is defined as

$$a \leq y \leq b, f_1(y) \leq x \leq f_2(y)$$

Here region R is bounded by the boundaries $y=a, y=b, x=f_1(y)$ & $x=f_2(y)$, Now to integrate we take a horizontal strip in R & integrate first w.r.t x (by keeping y as a constant) & then integrate w.r.t



y , so the integral becomes

$$\iint_R f(x,y) dx dy = \int_{y=a}^{y=b} \left[\int_{x=f_1(y)}^{x=f_2(y)} f(x,y) dx \right] dy$$



Ques: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} \, dx \, dy$

Solⁿ: Here $0 \leq y \leq a$ & $0 \leq x \leq \sqrt{a^2-y^2}$ so we integrate first w.r.t x .

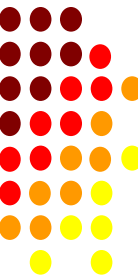
$$\begin{aligned} I &= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2} \, dx \right] dy \\ &= \int_0^a \left[\frac{x \sqrt{a^2-y^2-x^2}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy \end{aligned}$$

$$\int \sqrt{a^2-x^2} \, dx = \frac{x \sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1 \, dy = \frac{\pi}{4} \int_0^a (a^2-y^2) \, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right]$$

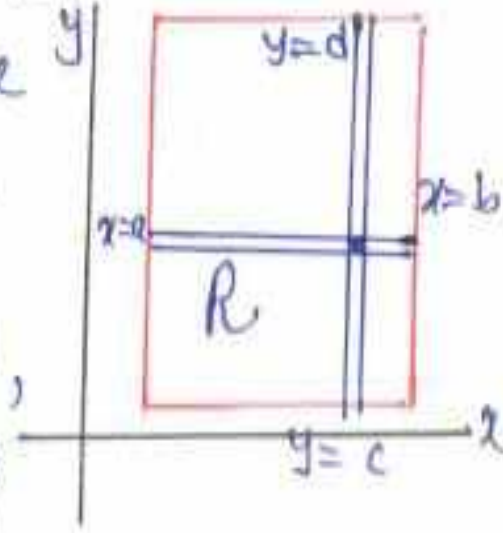
$$= \frac{\pi a^3}{6} \text{ Ans.}$$



Case: c $\Rightarrow$ If the Region R is defined as

$a \leq x \leq b$ & $c \leq y \leq d$ (where a, b, c & d are constants).

In this case the order of integration is immaterial, provided that the limits of integrations are changed accordingly.



Ques: Evaluate $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{1-x^2} \sqrt{1-y^2}}$

Solⁿ: Here $0 \leq x \leq 1$
 $0 \leq y \leq 1$ so both variables have
 constant limits.

$$\begin{aligned}
 I &= \int_0^1 \int_0^1 \frac{dx dy}{\sqrt{1-x^2} \sqrt{1-y^2}} \\
 &= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\int_0^1 \frac{1}{\sqrt{1-y^2}} dy \right] dx \\
 &= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\sin^{-1} y \right]_0^1 dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\frac{\pi}{2} - 0 \right] dx \\
 &= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} (\sin^{-1} x)_0^1 = \frac{\pi}{2} \times \frac{\pi}{2} \\
 &= \frac{\pi^2}{4} \quad \text{Ans.}
 \end{aligned}$$



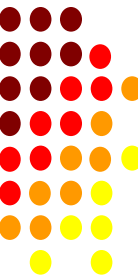
Ques: Evaluate $\int_1^2 \int_3^4 (xy + e^y) dy dx = \int_1^2 \left[\int_3^4 (xy + e^y) dy \right] dx$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y \right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3 \right) dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3 \right) dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x \right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3)$$

$$= \frac{21}{4} + e^4 - e^3 \text{ Ans}$$



Exp: Evaluate $\int_0^2 \int_0^1 (x^2 + 3y^2) dy dx$ [2019-20]

Solⁿ:
$$I = \int_0^2 \left[\int_0^1 (x^2 + 3y^2) dy \right] dx = \int_0^2 \left[x^2 y + \frac{3y^3}{3} \right]_0^1 dx$$

$$= \int_0^2 [x^2 + 1] dx = \left(\frac{x^3}{3} + x \right)_0^2 = \frac{8}{3} + 2$$

$$= \frac{8+6}{3} = \frac{14}{3} \text{ Ans.}$$



Exp: Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} dx dy$ [2018-19]

Solⁿ: Given $I = \int_0^1 \int_0^{x^2} e^{y/x} dy dx$

here $0 \leq y \leq x^2$ & $0 \leq x \leq 1$

$$I = \int_0^1 \left(\frac{e^{y/x}}{1/x} \right)_0^{x^2} dx = \int_0^1 x \left(e^{y/x} \right)_0^{x^2} dx$$

$$= \int_0^1 x [e^x - 1] dx$$

$$= \int_0^1 [x e^x - x] dx = \left[x e^x - \int 1 \cdot x e^x dx - \frac{x^2}{2} \right]_0^1$$

$$= \left[x e^x - e^x - \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2}$$

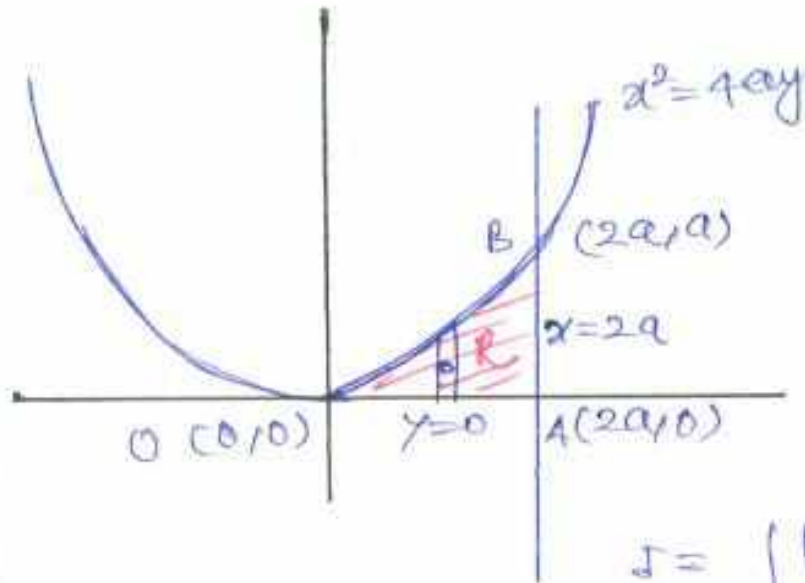
$$e^0 = 1 \text{ \& \; } e^1 = e$$



Exp: Find the value of the integral $\iint_R xy \, dx \, dy$, where R is the region bounded by the x -axis, the line $x=2a$ & the parabola $x^2=4ay$.

Sol: Here region R is the common part bounded by x -axis, $x=2a$ & $x^2=4ay$

[2011-12]



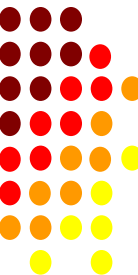
Intersection of $x^2=4ay$ & $x=2a$ is $(2a, a)$.

R is the region OAB , which is given by $0 \leq x \leq 2a$ &

$$0 \leq y \leq \frac{x^2}{4a}$$

so given integral is

$$\begin{aligned} I &= \iint_R xy \, dx \, dy = \int_{x=0}^{2a} \int_{y=0}^{\frac{x^2}{4a}} xy \, dy \, dx \\ &= \int_{x=0}^{2a} x \left(\frac{y^2}{2} \right)_0^{\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x^5}{32a^2} dx \\ &= \frac{1}{32a^2} \left(\frac{x^6}{6} \right)_0^{2a} = \frac{a^4}{3} \quad \text{Ans.} \end{aligned}$$

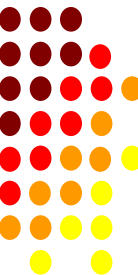
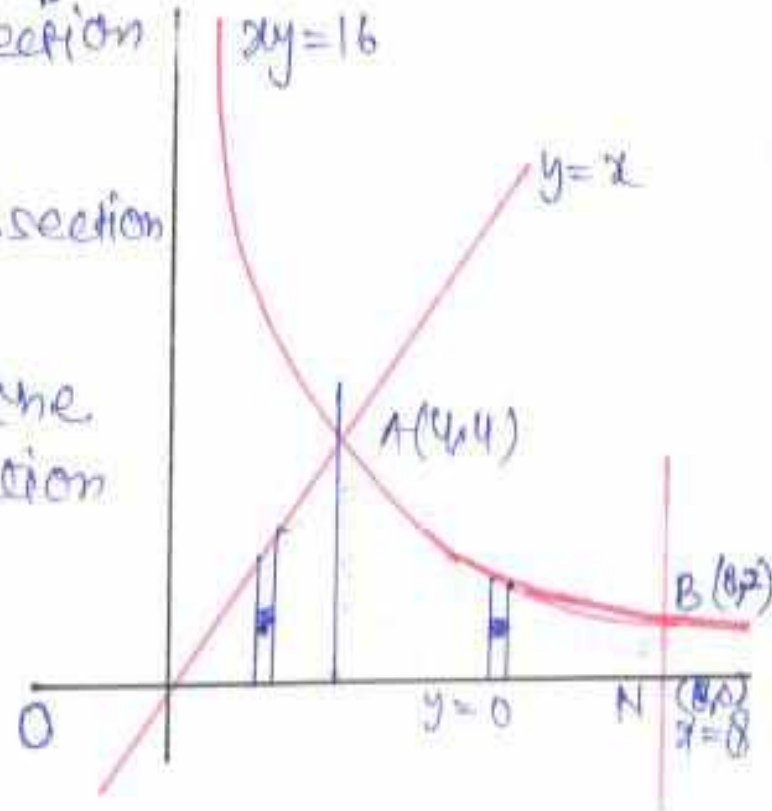


Exp: Let D be the region in the first quadrant bounded by the curves $xy=16$, $x=y$, $y=0$ & $x=8$. Sketch the region of integration of the following integral $\iint_D x^2 dx dy$ & evaluate it.

Solⁿ: $A(4,4)$ is the intersection of $xy=16$ & $y=x$.
 $B(8,2)$ is the intersection of $xy=16$ & $x=8$.
 Shaded portion is the region of integration

Given

$$I = \iint_D x^2 dx dy$$



To evaluate the given integral, we divide the area $OABNO$ into two parts by AM as shown in the figure.

$$\text{so } I = \iint_D x^2 dx dy = \iint_{OMAO} x^2 dx dy + \iint_{MNBAM} x^2 dx dy$$

$$= \int_{x=0}^4 \int_{y=0}^x x^2 dy dx + \int_{x=4}^8 \int_{y=0}^{\frac{16}{x}} x^2 dy dx$$

$$= \int_0^4 x^2 (y)_0^x dx + \int_4^8 x^2 (y)_0^{\frac{16}{x}} dx$$

$$= \int_0^4 x^3 dx + \int_4^8 16x dx = \left(\frac{x^4}{4} \right)_0^4 + \left(8x^2 \right)_4^8$$

$$= 64 + 8(64 - 16)$$

$$= 64 + 384$$

$$= 448 \quad \text{Ans.}$$



Ques No Evaluate $\int_0^a \int_{x/a}^x \frac{x}{x^2 + y^2} dy dx$

Sol: $I = \int_0^a x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_{x/a}^x dx$ $\left[\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$

$$= \int_0^a x \left[\frac{1}{x} \tan^{-1} 1 - \frac{1}{x} \tan^{-1} \frac{1}{a} \right] dx = \int_0^a \left(\frac{\pi}{4} - \tan^{-1} \frac{1}{a} \right) dx$$

$$= \left(\frac{\pi}{4} x - x \tan^{-1} \frac{1}{a} \right)_0^a = \frac{\pi}{4} a - a \tan^{-1} \frac{1}{a} \quad \text{Ans}$$



Ques No Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx dy$

Sol. $I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2} dx \right] dy$

$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$\left[\because \int \sqrt{a^2-x^2} dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1 dy = \frac{\pi}{4} \int_0^a (a^2-y^2) dy = \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right]$$

$$= \frac{\pi a^3}{6} \text{ Ans}$$



Practice Question

1. Evaluate $\int_0^1 \int_0^{x^2} x e^y dy dx$ [2017-10]

2. Evaluate $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}$

Ans 1:

Ans 2: $\frac{\pi^2}{4}$

3. Evaluate $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}} dy dx$

Ans 3: $e^2 - 1$



Practice Question

1. Evaluate: $\iint_R xy \, dx \, dy$ where R is the domain bounded by x -axis, ordinate $x = 2a$ and the curve

$$x^2 = 4ay$$

2. Evaluate: $\iint_R xy \, dx \, dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0, y \geq 0$

3. Evaluate $\int_1^0 \int_0^1 (x + y) \, dx \, dy$

Ans 1: $\frac{a^4}{3}$

4. Evaluate $\int_0^1 \int_0^x e^x \, dx \, dy$

Ans 2: $\frac{a^4}{8}$

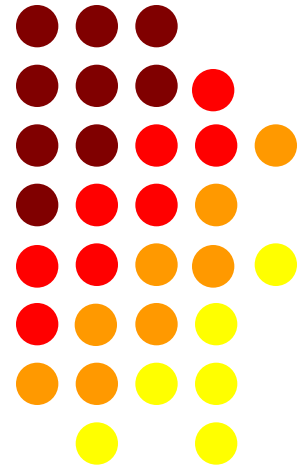
Ans 3: 1

Ans 4: 1



L-28

Double integral in Polar Coordinates



Introduction: Double integrals of the form

$$I = \int_{\theta=\theta_1}^{\theta_2} \int_{r=r_1}^{r_2} f(r, \theta) dr d\theta$$
 is known as

double integral in Polar coordinates. To solve these types of integrals we first solve w.r.t. r between the limits r_1 & r_2 & then solve w.r.t. θ between θ_1 to θ_2 .



Exp: Evaluate $I = \int_0^{2\pi} \int_{a \sin \theta}^a r dr d\theta$.

Sol.

$$\begin{aligned}
 I &= \int_0^{2\pi} \int_{r=a \sin \theta}^a r dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{r^2}{2} \right]_{a \sin \theta}^a d\theta = \frac{1}{2} \int_0^{2\pi} (a^2 - a^2 \sin^2 \theta) d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{a^2}{2} \int_0^{2\pi} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \\
 I &= \frac{a^2}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi} \\
 &= \frac{\pi a^2}{2}.
 \end{aligned}$$



Exp: Evaluate $\int_0^{\pi/2} \int_{a(1-\cos\theta)}^a r^2 dr d\theta$.

Sol: we have $I = \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta)}^a r^2 dr d\theta$.

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^3}{3} \right]_{r=a(1-\cos\theta)}^a d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{a^3}{3} - \frac{a^3(1-\cos\theta)^3}{3} \right] d\theta.$$

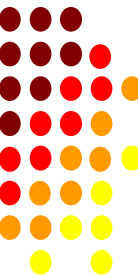
$$= \frac{a^3}{3} \int_{\theta=0}^{\pi/2} [1 - (1-\cos\theta)^3] d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} [1 - (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)] d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} [3\cos\theta - 3\cos^2\theta + \cos^3\theta] d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} \left[3\cos\theta - 3\left(\frac{1+\cos 2\theta}{2}\right) + \frac{1}{4}(\cos 3\theta + 3\cos\theta) \right] d\theta.$$

$$= \frac{a^3}{3} \left[3\sin\theta - \frac{3\theta}{2} - \frac{3}{2} \frac{\sin 2\theta}{2} + \frac{1}{4} \frac{\sin 3\theta}{3} + 3\sin\theta \right]_0^{\pi/2} = \frac{a^3}{36} [14 - 9\pi] \text{ Ans}$$

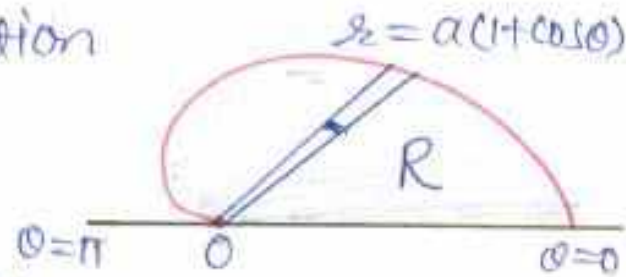


Exp: Evaluate $\iint_R r \sin \theta \, dr \, d\theta$ over the area of the cardioid $r = a(1 + \cos \theta)$ above the initial line.

Soln: Region 'R' of integration is given by

$$0 \leq \theta \leq \pi$$

$$\& \quad 0 \leq r \leq a(1 + \cos \theta).$$



$$I = \iint_R r \sin \theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos \theta)} r \sin \theta \, dr \, d\theta.$$

$$= \int_{\theta=0}^{\pi} \sin \theta \left(\frac{r^2}{2} \right)_0^{a(1+\cos \theta)} d\theta.$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi} \sin \theta \, a^2 (1 + \cos \theta)^2 d\theta.$$



putting $1 + \cos \theta = t \Rightarrow \sin \theta d\theta = -dt$

$$\theta = 0 \Rightarrow t = 2$$

$$\theta = \pi \Rightarrow t = 0$$

$$I = \frac{1}{2} \int_2^0 a^2 t^2 (-dt) = \frac{a^2}{2} \int_0^2 t^2 dt$$

$$= \frac{a^2}{2} \left(\frac{t^3}{3} \right)_0^2 = \frac{a^2}{2} \times \frac{8}{3}$$

$$= \frac{4a^2}{3} \text{ Ans.}$$



Exp: Evaluate $\iint r^3 dr d\theta$, over the area bounded between the circles $r=2\cos\theta$ & $r=4\cos\theta$.

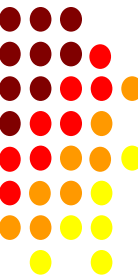
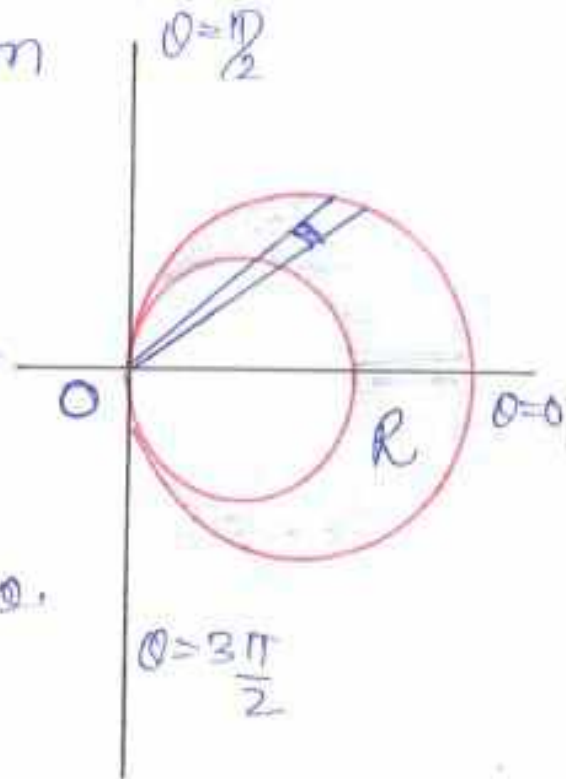
Solⁿ: Region R of integration is given by

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ \& }$$

$$2\cos\theta \leq r \leq 4\cos\theta, \quad \theta = \pi$$

$$I = \iint_R r^3 dr d\theta.$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=2\cos\theta}^{4\cos\theta} r^3 dr d\theta.$$



$$\begin{aligned}
 I &= \int_{-\pi/2}^{\pi/2} \left(\frac{2^4}{4} \right)^{4 \cos \theta} d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \frac{1}{4} [256 \cos^4 \theta - 16 \cos^4 \theta] d\theta \\
 &= 120 \int_0^{\pi/2} \cos^4 \theta d\theta \\
 &= 120 \times \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = \frac{45}{2} \pi
 \end{aligned}$$

formula $\int_0^{\pi/2} \cos^n \theta d\theta = \int_0^{\pi/2} \sin^n \theta d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2}$

if n is even

$$\int_0^{\pi/2} \cos^n \theta d\theta = \int_0^{\pi/2} \sin^n \theta d\theta = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}, \text{ if } n \text{ is odd}$$



Practice Question

Evaluate the following:

1. $\iint r e^{\frac{-r^2}{a^2}} \cos \theta \sin \theta \, dr \, d\theta$, over the upper half of the circle $r = 2a \cos \theta$.

$$\left[\text{Ans. : } \frac{a^2}{16} \left(3 + \frac{1}{e^4} \right) \right]$$

2. $\iint r^3 \, dr \, d\theta$, over the region between the circles $r = 2 \sin \theta$ and $r = 4 \sin \theta$.

$$\left[\text{Ans. : } \frac{45\pi}{2} \right]$$

3. $\iint r \sin \theta \, dA$, over the cardioid $r = a (1 + \cos \theta)$ above the initial line.

$$\left[\text{Ans. : } \frac{4}{3} a^3 \right]$$

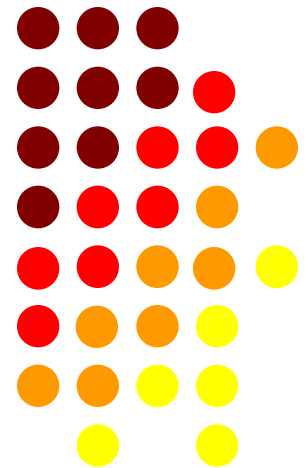
4. $\iint \frac{r}{\sqrt{r^2 + 4}} \, dr \, d\theta$, over one loop of the lemniscate $r^2 = 4 \cos 2\theta$.

$$\left[\text{Ans. : } (4 - \pi) \right]$$



L-29

Change of order of integration



Introduction: On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration. Value of integration is unchanged by changing of the order of integration.

Some complicated integrals can be made easy to handle by a change in the order of integration.

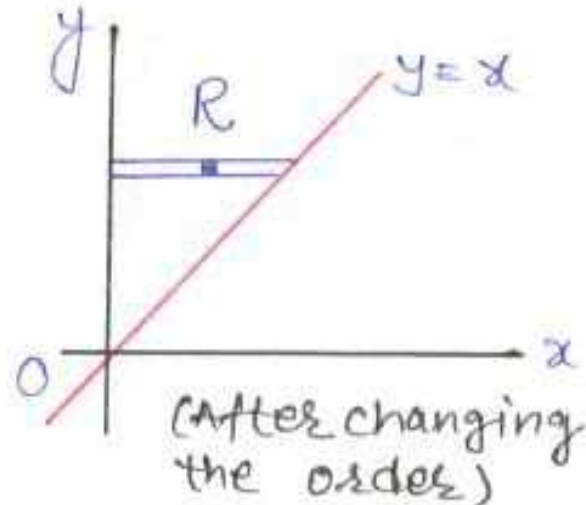
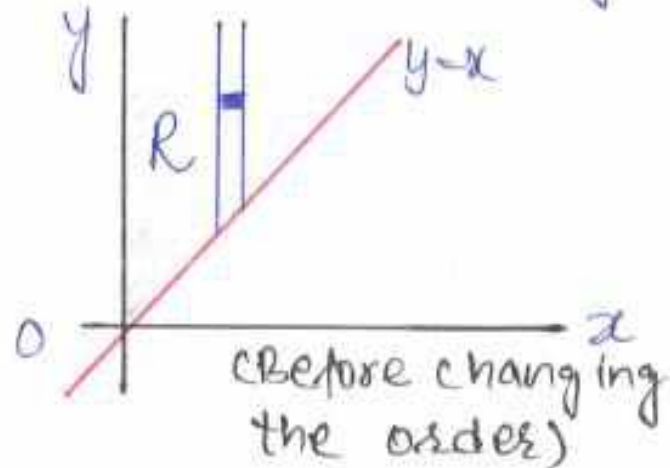


Exp: Evaluate the following integral by changing the order of integration, $\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dy dx$ [2020-21]

Solⁿ: Given integral is $I = \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dy dx$

Given region of integration is

$$0 \leq x < \infty \text{ \& \> } x \leq y < \infty$$



After changing of the order of integration R can be written as $0 \leq x \leq y, 0 \leq y < \infty$



$$\text{So } I = \int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dy dx \quad (\text{Before})$$

$$= \int_0^{\infty} \int_0^y \frac{e^{-y}}{y} dx dy \quad (\text{After})$$

$$= \int_0^{\infty} \left(\frac{e^{-y}}{y} \right) \int_0^y (1) dx dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y} (x)_0^y dy = \int_0^{\infty} \frac{e^{-y}}{y} \times y dy$$

$$= \left[\frac{e^{-y}}{-1} \right]_0^{\infty} = \left(-\frac{1}{e^y} \right)_0^{\infty} = \left[\frac{1}{\infty} - 1 \right] = -(0-1)$$

$$= 1 \text{ Ans.}$$



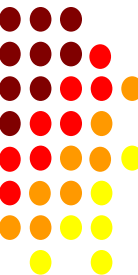
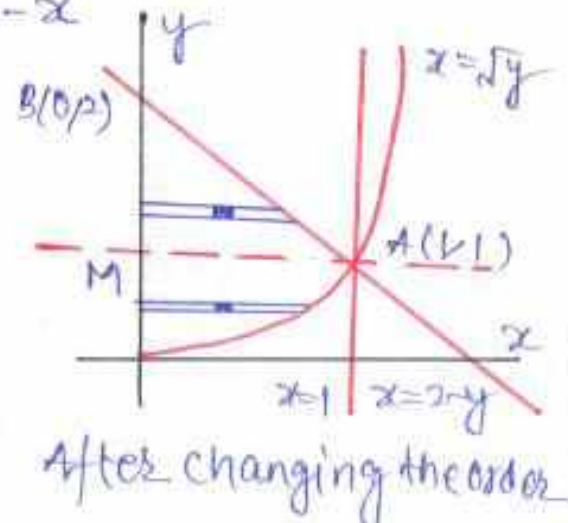
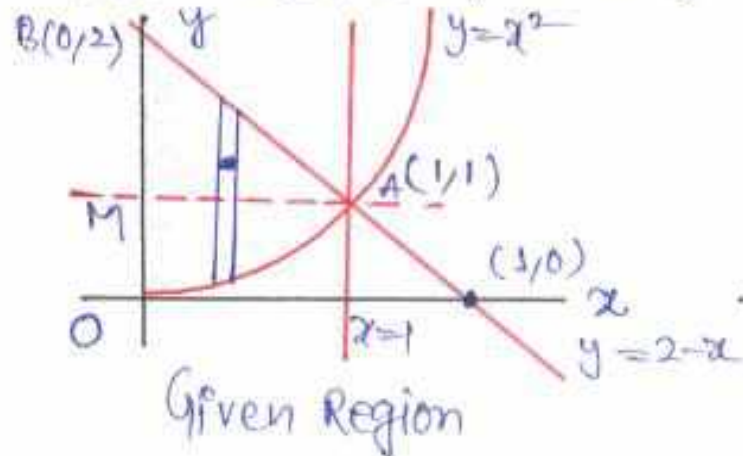
Exp: Evaluate the integration by changing of order of $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

[2014-15, 2015-16, 2016-17, 2017-18, 2019-20]

Solⁿ: Given $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ — ①

where region of integral is bounded by the curves $y=x^2, y=2-x, x=0$ & $x=1$

i.e. $0 \leq x \leq 1, x^2 \leq y \leq 2-x$



After changing the order we have

$$\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx = \iint_{\text{OAMO}} xy \, dx \, dy + \iint_{\text{MABM}} xy \, dx \, dy$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy + \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$= \frac{1}{2} \int_0^1 y (x^2)_0^{\sqrt{y}} dy + \int_1^2 y \left(\frac{x^2}{2} \right)_0^{2-y} dy$$

$$= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y (2-y)^2 dy$$

$$= \frac{1}{2} \left(\frac{y^3}{3} \right)_0^1 + \frac{1}{2} \int_1^2 [4y - 4y^2 + y^3] dy$$

$$= \frac{1}{6} + \frac{1}{2} \left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4} \right]_1^2$$

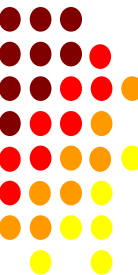
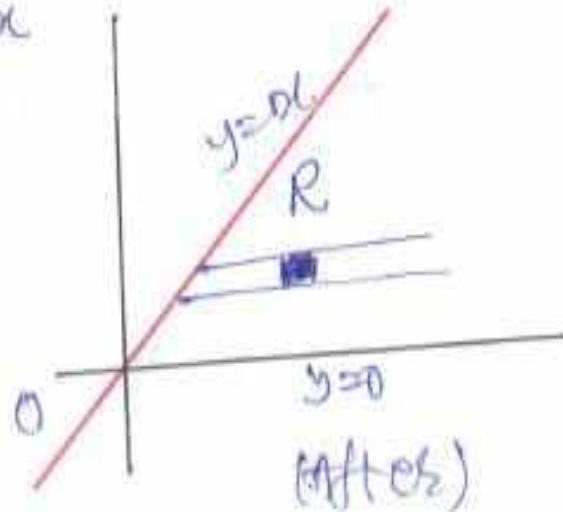
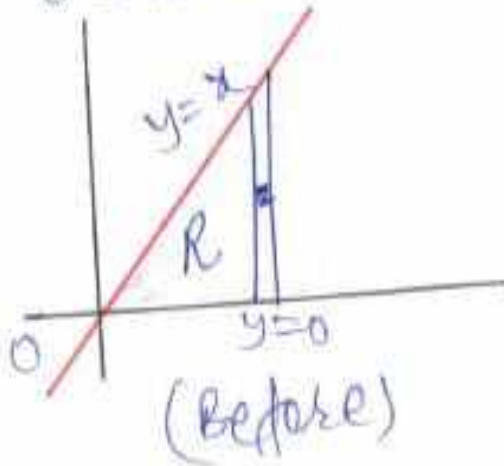
$$= \frac{1}{6} + \frac{1}{2} \left[\left(8 - \frac{32}{3} + 4 \right) - \left(2 - \frac{4}{3} + \frac{1}{4} \right) \right]$$

$$= \frac{3}{2} \quad \text{Ans.}$$



Exp: Evaluate the integral $\int_0^\infty \int_0^x x \cdot \exp\left(-\frac{x^2}{y}\right) dy dx$
by changing the order of integration.

Soln: Region of given integral is bounded by
 $0 \leq x < \infty$, $0 \leq y \leq x$



After changing the order, region is given by

$$y \leq x < \infty, \quad 0 \leq y < \infty$$

$$I = \int_0^{\infty} \int_0^x x \cdot \exp\left(-\frac{x^2}{y}\right) dy dx \quad (\text{Before})$$

$$= \int_0^{\infty} \int_y^{\infty} x \cdot e^{-x^2/y} dx dy \quad (\text{After})$$

$$= \int_0^{\infty} \left[\int_y^{\infty} x e^{-x^2/y} dx \right] dy$$

$$= \int_0^{\infty} \left[-\frac{y}{2} e^{-x^2/y} \right]_y^{\infty} dy = \int_0^{\infty} \frac{y}{2} e^{-y} dy$$

$$= \left[\frac{y}{2} (-e^{-y}) - \frac{1}{2} (e^{-y}) \right]_0^{\infty} = [(0-0) - (0-\frac{1}{2})]$$

$$= \frac{1}{2} \text{ Ans.}$$



Exp Change the order of integration and evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy \, dy \, dx$.

Solution

1. Since inner limits depend on x , the function is integrated first w.r.t. y .

2. Limits of $y: y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$,

along vertical strip $A'B'$

3. The region is bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

4. The points of intersection of $x^2 = 4ay$ and $y^2 = 4ax$ are obtained as

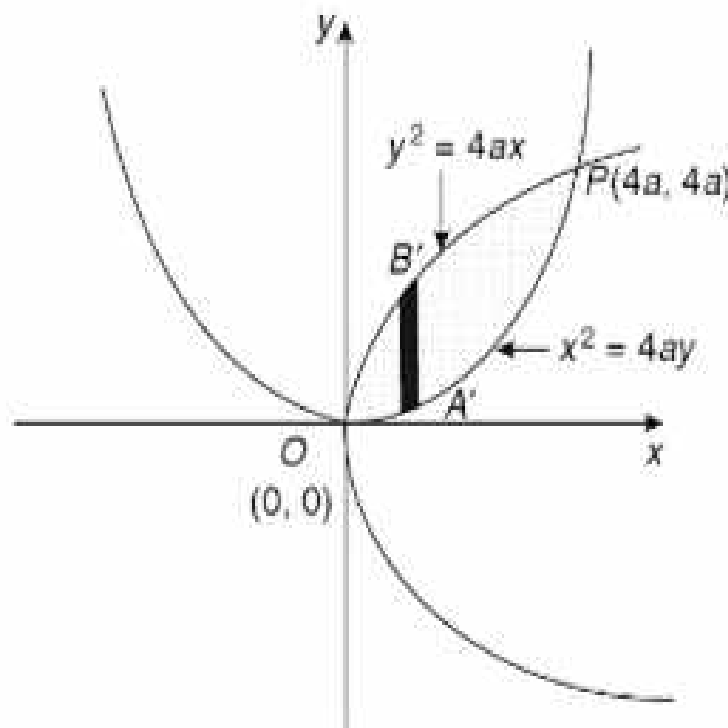
$$\begin{aligned} x^4 &= 16a^2y^2 \\ &= 16a^2(4ax) \end{aligned}$$

$$x(x^3 - 64a^3) = 0$$

$$x = 0, x = 4a$$

$$\therefore y = 0, y = 4a$$

The points of intersection are $O(0, 0)$ and $P(4a, 4a)$.



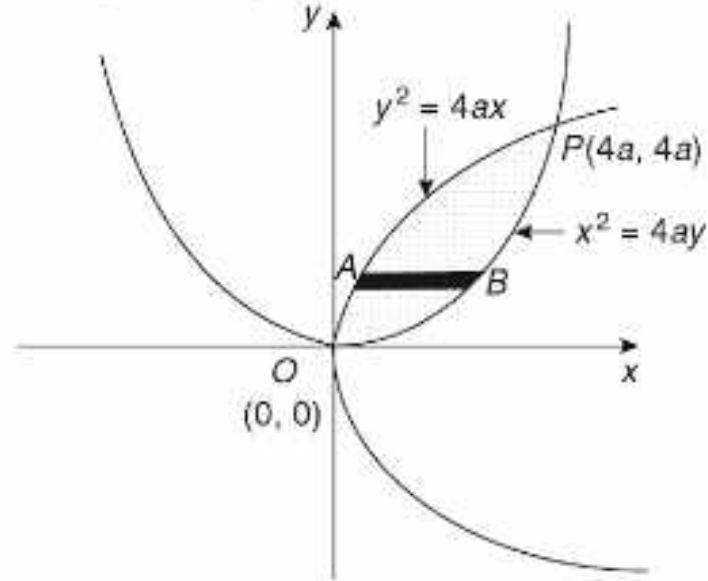
5. To change the order of integration, i.e., to integrate first w.r.t. x , draw a horizontal strip AB parallel to x -axis which starts from the parabola $y^2 = 4ax$ and terminates on the parabola $x^2 = 4ay$.

Limits of x : $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

Limits of y : $y = 0$ to $y = 4a$

Hence, the given integral after change of order is

$$\begin{aligned} \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy \, dy \, dx &= \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy \, dx \, dy \\ &= \int_0^{4a} \left[\frac{x^2}{2} \right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} y \, dy \\ &= \frac{1}{2} \int_0^{4a} \left(4ay - \frac{y^4}{16a^2} \right) dy \\ &= \frac{1}{2} \left[4a \cdot \frac{y^2}{2} - \frac{1}{16a^2} \cdot \frac{y^5}{5} \right]_0^{4a} \\ &= \frac{1}{2} \left[2a(16a^2) - \frac{1}{16a^2} \cdot \frac{(4a)^5}{5} \right] \\ &= \frac{1}{2} \cdot \frac{96}{5} a^3 \\ &= \frac{48}{5} a^3 \end{aligned}$$



Practice questions

Ques 1: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(\sqrt{1-x^2-y^2})(e^y+1)} dx dy$ [2011-12]

Ques 2: changing the order of integration in
 $I = \int_0^\infty \int_{x/4}^2 f(x,y) dy dx$ leads to [2016-17]

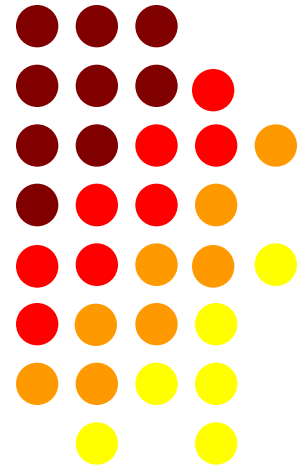
$$I = \int_2^8 \int_p^q f(x,y) dx dy, \text{ say, what is } q?$$

Ques 3: change the order of integration & evaluate $\int_0^2 \int_{x^2/4}^{3-x} xy dy dx$ [2018-19]



L-30

Area by Double Integral



Introduction: Area by double integration is

given by

a) Area of region $R = \iint_R dx dy$ [in Cartesian]

b) Area of region $R = \iint_R r dr d\theta$ [in Polar]

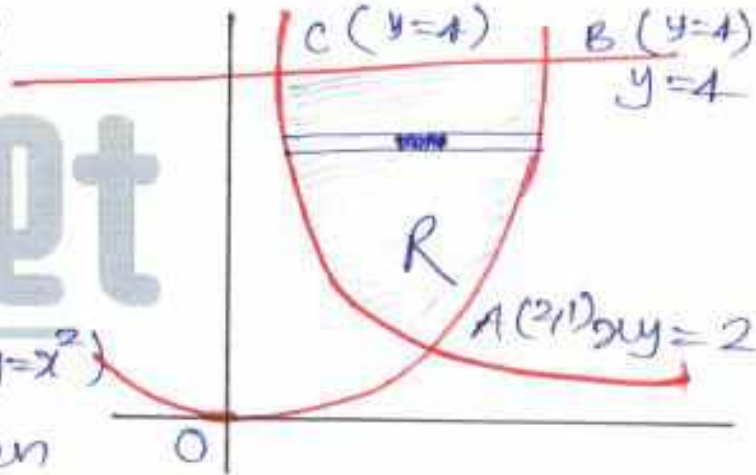


Exp: Determine the area bounded by the curves $xy = 2$, $4y = x^2$ & $y = 4$ in 1st quadrant. [2014-15]

Solⁿ: Required area of shaded region is

$$A = \iint_R dx dy \quad \text{--- (1)}$$

$(4y = x^2)$



where R is given

$$\text{by } \Rightarrow 1 \leq y \leq 4, \quad \frac{2}{y} \leq x \leq 2\sqrt{y}$$

$$A = \int_{y=1}^4 \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx dy = \int_1^4 \left[2\sqrt{y} - \frac{2}{y} \right] dy$$

$$= 2 \left[\frac{2}{3} y^{3/2} - \log y \right]_1^4$$

$$= 2 \left[\frac{16}{3} - 2\log 2 - \frac{2}{3} \right]$$

$$= \frac{28}{3} - 4\log 2 \quad (\text{Ans}).$$



Exp: Compute the area of lemniscate [2013-14]

$$r^2 = a^2 \cos 2\theta$$

Solⁿ: Given curve is

$$r^2 = a^2 \cos 2\theta$$

$$\text{or } r = \pm a \sqrt{\cos 2\theta}$$

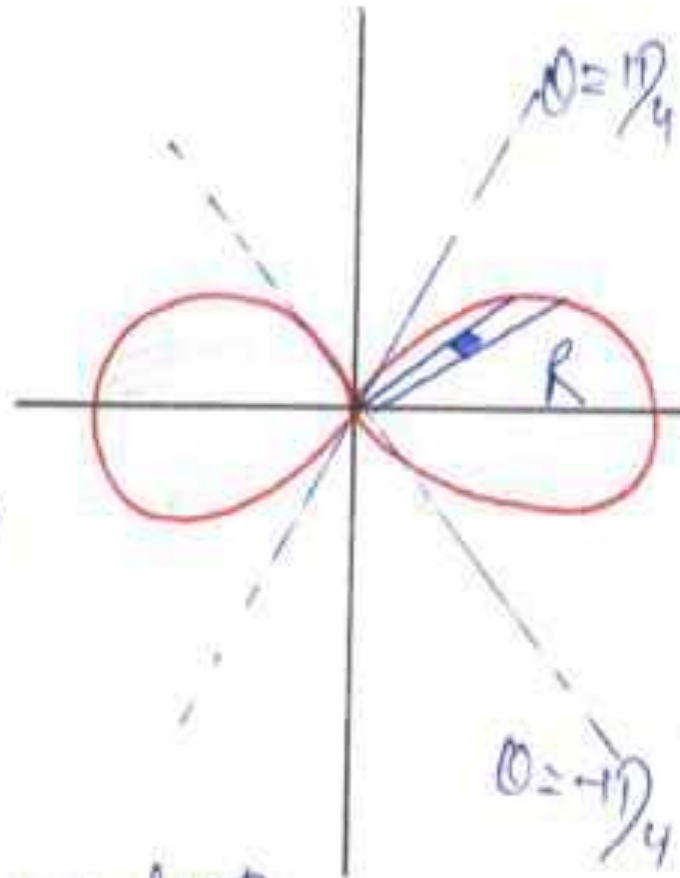
here θ lies between

$$\theta = 0 \text{ to } \theta = \frac{\pi}{4}$$

Required area

$$A = 4 \times \text{Area of } R \text{ in one loop}$$

$$= 4 \iint_R r \, dr \, d\theta$$



$$A = 4 \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r \, dr \, d\theta.$$

$$= 4 \int_0^{\pi/4} \left(\frac{r^2}{2} \right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 4 \times \frac{1}{2} \int_0^{\pi/4} (a\sqrt{\cos 2\theta})^2 d\theta.$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta d\theta$$

$$= 2a^2 \left(\frac{\sin 2\theta}{2} \right)_0^{\pi/4}$$

$$= \frac{2a^2}{2} [\sin \pi/2 - 0]$$

$$= a^2 \quad \text{Ans.}$$



Exp: Evaluate the area enclosed between the parabola $y=x^2$ & the straight line $y=x$.

Solⁿ: Area of the shaded region

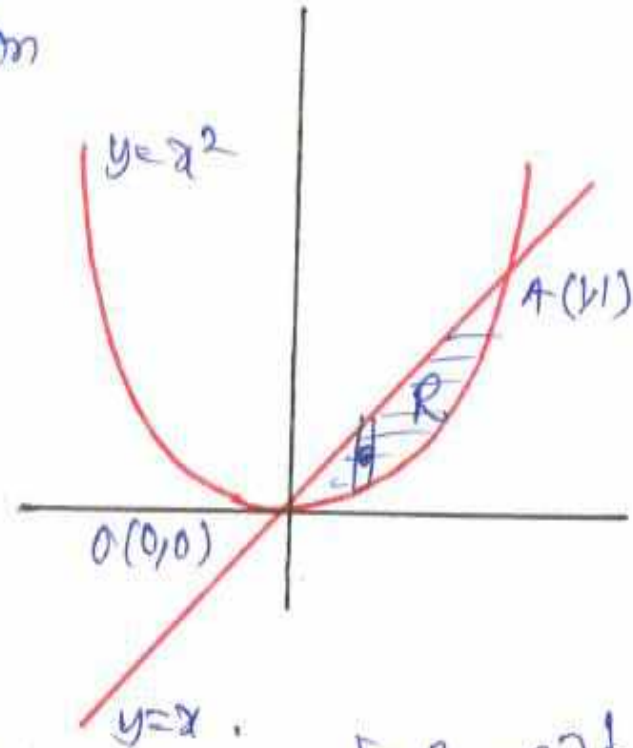
$$i) A = \iint_R dx dy$$

$$= \int_{x=0}^1 \int_{y=x^2}^x dy dx$$

$$= \int_0^1 [y]_{x^2}^x dx$$

$$= \int_0^1 [x - x^2] dx = \int_0^1 [x - x^2] dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ Ans.}$$



Exp: find the area outside the circle $r=a$ & inside the cardioid $r=a[1+\cos\theta]$

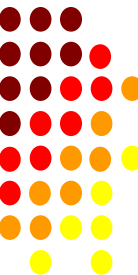
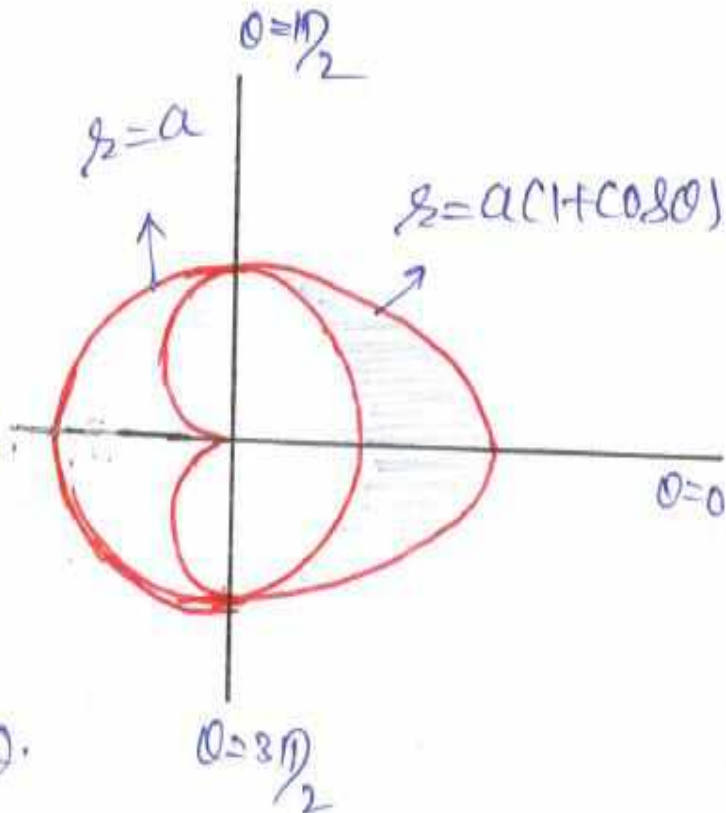
Solⁿ: Area of the shaded region is

$$A = \iint_R r \, dr \, d\theta$$

$$= 2 \int_0^{\pi/2} \int_{r=a}^{r=a(1+\cos\theta)} r \, dr \, d\theta$$

$$= 2 \int_0^{\pi/2} \left[\frac{a^2(1+\cos\theta)^2}{2} - \frac{a^2}{2} \right] d\theta$$

$$= a^2 \int_0^{\pi/2} [1 + 2\cos\theta + \cos^2\theta - 1] d\theta$$



$$\text{Area} = a^2 \int_0^{\pi/2} \left[2\cos\theta + \left(\frac{1+\cos 2\theta}{2} \right) \right] d\theta.$$

$$= \frac{a^2}{2} \int_0^{\pi/2} [1 + 4\cos\theta + \cos 2\theta] d\theta.$$

$$= \frac{a^2}{2} \left[\theta + 4\sin\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{2} + 4 \right] = \frac{a^2}{4} (\pi + 8) \text{ Ans.}$$



Ques :

Example 1. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

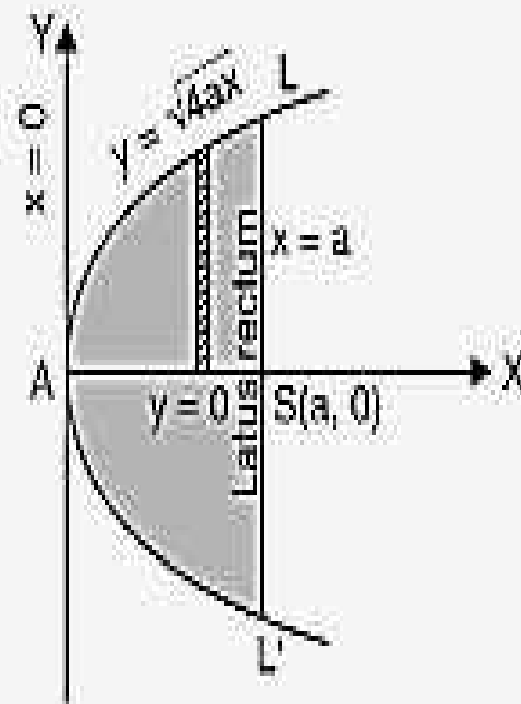
Solution. Required area = 2 (area (ASL))

$$= 2 \int_0^a \int_0^{2\sqrt{ax}} dy dx$$

$$= 2 \int_0^a 2\sqrt{ax} dx$$

$$= 4\sqrt{a} \left(\frac{x^{3/2}}{3/2} \right)_0^a = \frac{8a^2}{3}$$

Ans.



HOME WORK QUESTIONS

1) Determine the area of the region bounded by the curves $xy = 2$, $4y = x^2$, $y = 4$.

(Uptu-2001,2008) **Ans** = $\frac{28}{3} - 4 \log 2$

2) Find by double integration the area bounded by the pair of axis $y = 2 - x$ and $y^2 = 2(2 - x)$. **Ans** = $\frac{2}{3}$

Question 3. Determine the area of region bounded by the curves

$$xy = 2, 4y = x^2, y = 4.$$

Answer $\frac{28}{3} - 4 \log 2.$

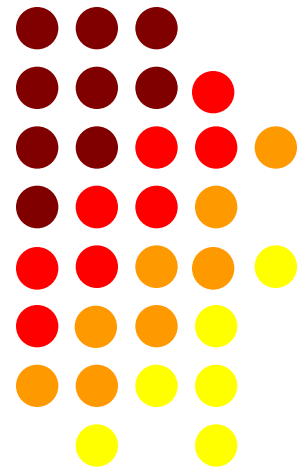
Question 4: Find the area inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = 2a \cos \theta$.

Ans: $\pi \frac{a^2}{2}$



L-31

Introduction to Triple integration, volume by triple integral



Introduction to Triple integration

Introduction: Triple integral of a function of three variables $f(x, y, z)$ in a region V is denoted by $I = \iiint_V f(x, y, z) dx dy dz$



Evaluation of Triple integral: If region V is defined by $a \leq x \leq b$, $c \leq y \leq d$, $g \leq z \leq h$ where a, b, c, d, g, h all are constants.

then
$$I = \int_{x=a}^b \int_{y=c}^d \int_{z=g}^h f(x, y, z) \, dz \, dy \, dx$$

can be calculated first w.r.t z between g and h , then w.r.t y between c and d & then w.r.t x between a and b .



If V is given by
 $a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)$ &

$f_1(x, y) \leq z \leq f_2(x, y)$ then

$$I = \int_{x=a}^b \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x, y, z) dz dy dx$$

If z is the inner most integral then we solve the above integral first w.r.t z by keeping x, y as constant then w.r.t y by keeping x as a constant & at last w.r.t x between a to b .



Exp: Evaluate $\iiint (x+y+z) dx dy dz$, where

R !, $0 \leq x \leq 1, 1 \leq y \leq 2; 2 \leq z \leq 3$ [2015-16, 2017-18]

Solⁿ: $I = \int_{x=0}^1 \int_{y=1}^2 \int_{z=2}^3 (x+y+z) dz dy dx$

$$= \int_{x=0}^1 \int_{y=0}^2 \left[(x+y)z + \frac{z^2}{2} \right]_2^3 dy dx$$

$$= \int_0^1 \int_{y=0}^2 \left[(x+y) + \frac{5}{2} \right] dy dx$$

$$= \int_0^1 \left[\left(x + \frac{5}{2}\right) y + \frac{y^2}{2} \right]_0^2 dx = \int_0^1 \left[\left(x + \frac{5}{2}\right) + \frac{3}{2} \right] dx$$

$$= \left(\frac{x^2}{2} + 4x \right)_0^1 = \frac{9}{2} \text{ Ans.}$$



Example 2 Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z} dx dy dz$.

Solution.
$$I = \int_0^{\log 2} \int_0^x e^{x+y} \left[e^z \right]_0^{x+y} dx dy$$

$$= \int_0^{\log 2} \int_0^x e^{x+y} (e^{x+y} - 1) dx dy = \int_0^{\log 2} \int_0^x [e^{2(x+y)} - e^{(x+y)}] dx dy$$

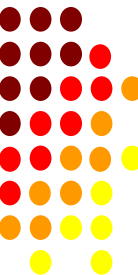
$$= \int_0^{\log 2} \left[e^{2x} \cdot \frac{e^{2y}}{2} - e^x \cdot e^y \right]_0^x dx = \int_0^{\log 2} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x \right]_0^{\log 2} = \left[\frac{e^{4 \log 2}}{8} - \frac{e^{2 \log 2}}{2} - \frac{e^{2 \log 2}}{4} + e^{\log 2} \right] - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right)$$

$$= \left(\frac{e^{\log 16}}{8} - \frac{e^{\log 4}}{2} - \frac{e^{\log 4}}{4} + e^{\log 2} \right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right)$$

$$= \left(\frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2 \right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right) = \frac{5}{8}$$

Ans.



Volume by Triple integration

Volume of the solid 'R' is given by

$$\text{Volume} = \iiint_R dx dy dz$$



Exp: find the volume of the solid bounded by the surfaces $x=0$, $y=0$, $z=0$ & $x+y+z=1$
[2018-19, 2020-21]

Solⁿ: Volume = $\iiint_R dx dy dz$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} (z)_0^{1-x-y} dy dx$$

$$= \int_0^1 \int_{y=0}^{1-x} (1-x-y) dy dx$$

$$= \int_0^1 \left[(-x+1)y - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 [(1-x)^2] dx$$

$$= \frac{1}{2} \left[\frac{(1-x)^3}{-3} \right]_0^1 = \frac{1}{6} \text{ Ans.}$$



Exp: find the volume of the region bounded by the surfaces $y=x^2$, $x=y^2$ & the planes $z=0$, $z=3$ [2019-20]

Solⁿ: Volume = $\iiint_R dx dy dz$ — ①

$$= \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^3 dz dy dx$$

$$= \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} [z]_0^3 dy dx = 3 \int_0^1 (y)_{x^2}^{\sqrt{x}} dx$$

$$= 3 \int_0^1 [\sqrt{x} - x^2] dx$$

$$= 3 \int_0^1 \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right) dx$$

$$= 3 \left(\frac{2}{3} - \frac{1}{3} \right)$$

$$= 3 \left[\frac{1}{3} \right]$$

$$= 1 \text{ cubic unit}$$



Example A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the plane $z = 0$ and surface $z = c + xy$.

Sol. Here x varies from 0 to a

y varies from 0 to $\frac{bx}{a}$

z varies from 0 to $c + xy$

Hence, the volume is

$$\begin{aligned}
 V &= \int_0^a \int_0^{bx/a} \int_0^{c+xy} dz \, dy \, dx = \int_0^a \int_0^{bx/a} (c + xy) \, dy \, dx \\
 &= \int_0^a \left[cy + \frac{xy^2}{2} \right]_0^{bx/a} dx = \int_0^a \left(\frac{bcx}{a} + \frac{b^2 x^3}{2a^2} \right) dx \\
 &= \left[\frac{bcx^2}{2a} + \frac{b^2 x^4}{8a^2} \right]_0^a = \frac{bca^2}{2a} + \frac{b^2 a^4}{8a^2} = \frac{ab}{8} (4c + ab).
 \end{aligned}$$



Practice questions

Ques: Evaluate the triple integral
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$$

[2016-17]

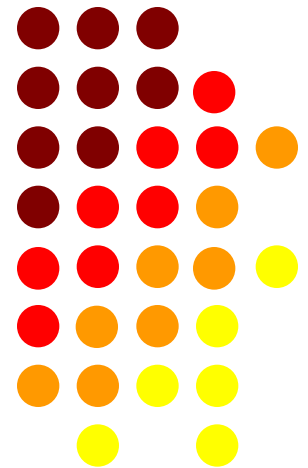
Ques: Evaluate $\iiint x^2 y z \, dz \, dy \, dx$ throughout
the volume bounded by the planes
 $x=0, y=0, z=0$ & $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
[2016-17]

Ques: find the volume of the solid which
is bounded by the surfaces
 $2z = x^2 + y^2$ & $z = x$ [2011-12]



L-32

Change of variable in double and triple integral



Change of variable in double integral

Let the double integral

$$I = \iint_R f(x, y) dx dy \text{ - (1) if it is to be changed}$$

in the new variables u & v .

Relation between u, v, x & y is given by

$$x = \phi(u, v), \quad y = \psi(u, v)$$

Then

$$I = \iint_R f(x, y) dx dy$$

$$= \iint_{R'} f[\phi(u, v), \psi(u, v)] |J| du dv \text{ - (2)}$$

$$\text{where } dx dy = |J| du dv \quad \& \quad J = \frac{\partial(x, y)}{\partial(u, v)}$$



Change of variables from (x, y) to Polar
coordinates (r, θ) $\Rightarrow$

$$\text{Here } x = r \cos \theta, \quad y = r \sin \theta$$

$$\iint_R f(x, y) \, dx \, dy = \iint f[r \cos \theta, r \sin \theta] r \, dr \, d\theta$$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r \quad \text{so } dx \, dy = r \, dr \, d\theta.$$

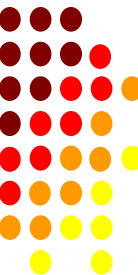


Exp: Evaluate $\iint_R (x+y)^2 dx dy$, where R is the region bounded by the parallelogram in the xy -plane with vertices $(1,0)$, $(3,1)$, $(2,2)$, $(0,1)$, using the transformation

$$u = x+y, v = x-2y \quad [2019-20]$$

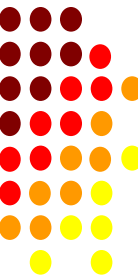
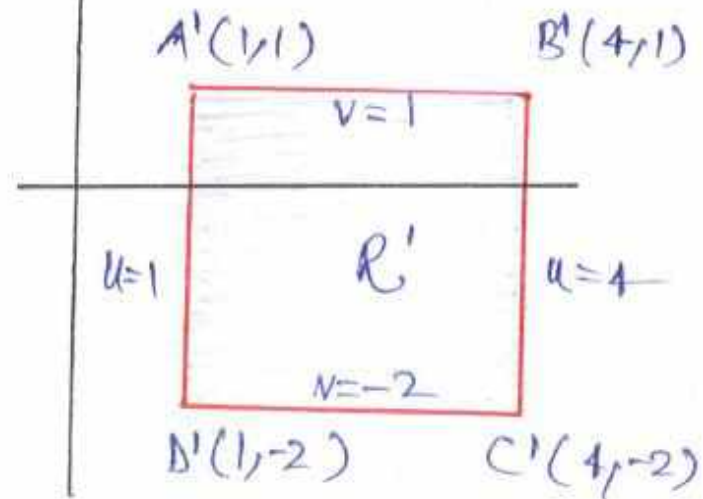
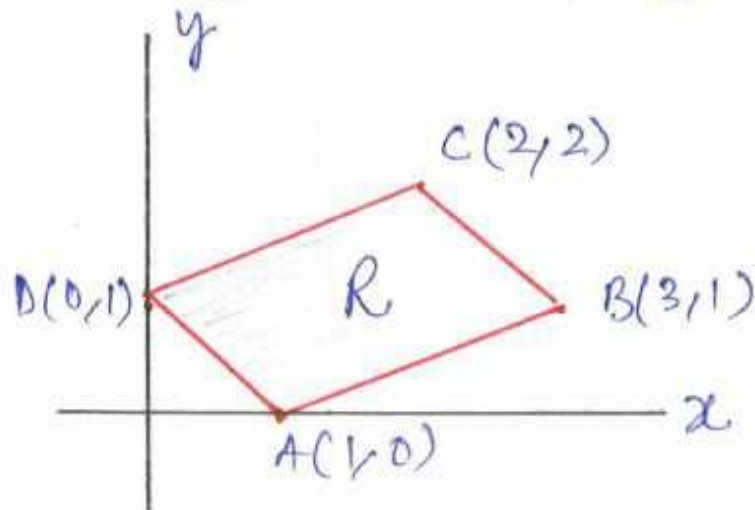
Solⁿ: The vertices $A(1,0)$, $B(3,1)$, $C(2,2)$, $D(0,1)$ of the parallelogram $ABCD$ in xy -plane become $A'(1,1)$, $B'(4,1)$, $C'(4,-2)$ & $D'(1,-2)$ in the uv -plane by using the transformations

$$u = x+y \text{ \& } v = x-2y$$



The region R in xy plane becomes the region R' in the uv -plane which is a rectangle bounded by the line $u=1$, $u=4$, $v=-2$ & $v=1$. Solving the given equations for x & y we get

$$x = \frac{1}{3}(2u+v), \quad y = \frac{1}{3}(u-v).$$



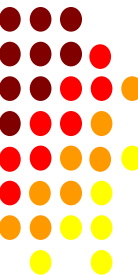
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$dx dy = |J| du dv = \left| -\frac{1}{3} \right| du dv = \frac{1}{3} du dv$$

$$\iint_R (x+y)^2 dx dy = \iint_{R'} u^2 |J| du dv$$

$$= \int_{-2}^1 \int_1^4 \frac{u^2}{3} du dv = \int_{-2}^1 \frac{1}{3} \left(\frac{u^3}{3} \right)_1^4 dv$$

$$= \int_{-2}^1 7 dv = 7 \times 3 = 21 \text{ Ans.}$$



Exp: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ by changing to polar coordinates, hence show that

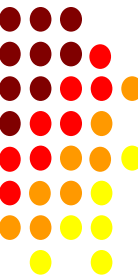
$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad [2018-19]$$

Solⁿ: Given region of integration is 1st quadrant. To change it into polar we have

$$x = r \cos \theta, \quad y = r \sin \theta.$$

$$dx dy = r dr d\theta.$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy dx = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta.$$



$$= \int_0^{\infty} \int_0^{\infty} e^{-z^2} z \, dz \, d\theta.$$

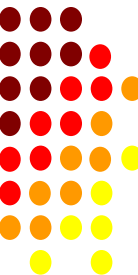
$$= \int_0^{\infty} \int_0^{\infty} \frac{e^{-t}}{2} \, dt \, d\theta, \text{ where } t = z^2$$

$$= \int_0^{\infty} \left[-\frac{1}{2} e^{-t} \right]_0^{\infty} d\theta.$$

$$= -\frac{1}{2} \int_0^{\infty} (0 - 1) d\theta.$$

$$= \frac{1}{2} (\theta)_0^{\infty}$$

$$= \frac{\pi}{4}$$



now let $I = \int_0^{\infty} e^{-x^2} dx$

between the same limits, we have

$$I = \int_0^{\infty} e^{-y^2} dy$$

$$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-x^2} \cdot e^{-y^2} dx dy$$

$$= \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$



Ques 3.

Using the transformation $x + y = u$, $y = uv$; show that

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)} dy dx = \frac{1}{2} (e - 1)$$

Sol. Since

$$x = u(1 - v), y = uv$$

$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\therefore dx dy = |J| du dv = u du dv$$

$$\text{Also, } x = 0 \Rightarrow u(1 - v) = 0 \Rightarrow u = 0, v = 1$$

$$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0, v = 0$$

$$x + y = 1 \Rightarrow u = 1$$

Hence the limits of u are 0 to 1 and the limits of v are 0 to 1.

$$\begin{aligned} \therefore \int_0^1 \int_0^{1-x} e^{y/(x+y)} dy dx &= \int_0^1 \int_0^1 e^{uv/u} |J| du dv \\ &= \int_0^1 \int_0^1 u e^v du dv = \left(\frac{u^2}{2} \right)_0^1 \left(e^v \right)_0^1 = \frac{1}{2} (e - 1). \end{aligned}$$



Example 4 . Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2) dy dx$

Solution. $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2) dy dx$

Limits of $y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0 \quad \dots(1)$

(1) represents a circle whose centre is (1, 0) and radius = 1.

Lower limit of y is 0 i.e., x -axis.

Region of integration is upper half circle.

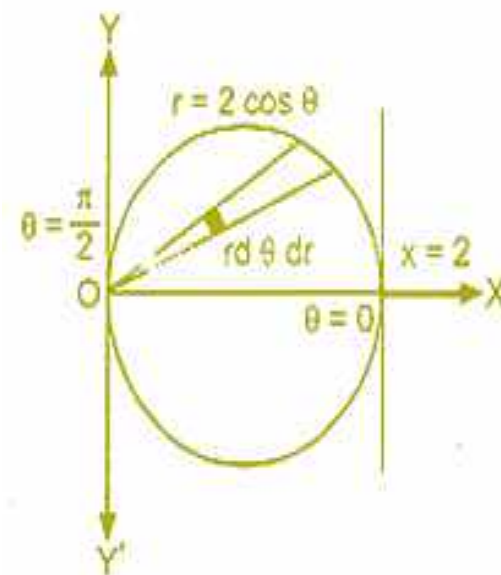
Let us convert (1) into polar co-ordinates by putting

$x = r \cos \theta, y = r \sin \theta$

$r^2 - 2r \cos \theta = 0 \Rightarrow r = 2 \cos \theta$

Limits of r are 0 to $2 \cos \theta$

Limits of θ are 0 to $\frac{\pi}{2}$



$$\begin{aligned}
 & \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2) dy dx \\
 &= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 (r d\theta dr) \\
 &= \int_0^{\pi/2} d\theta \int_0^{2\cos\theta} r^3 dr \\
 &= \int_0^{\pi/2} d\theta \left[\frac{r^4}{4} \right]_0^{2\cos\theta} \\
 &= 4 \int_0^{\pi/2} \cos^4 \theta d\theta \\
 &= 4 \times \frac{3 \times 1 \times \pi}{4 \times 2 \times 2} \\
 &= \frac{3\pi}{4} \text{ Ans.}
 \end{aligned}$$



Exp: Evaluate the following by changing into polar coordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 \sqrt{x^2+y^2} dx dy$

Solⁿ: changing to polar coordinates

we have $x = r \cos \theta$, $y = r \sin \theta$

$$\text{so } x^2 + y^2 = r^2 \Rightarrow r^2 = a^2$$

$$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 \sqrt{x^2+y^2} dx dy$$

$$= \int_0^{\pi/2} \int_0^a r^2 \sin^2 \theta \cdot r \cdot r dr d\theta$$

$$= \int_0^{\pi/2} \sin^2 \theta \cdot \left(\frac{r^5}{5} \right)_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \left(1 - \frac{\cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^5}{10} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}$$



Change of variables in triple integral

$I = \iiint_V f(x, y, z) dx dy dz$, can be changed
in to the variables u, v, w as

$$I = \iiint_{V'} f[\phi_1(x, y, z), \phi_2(x, y, z), \phi_3(x, y, z)] |J| du dv dw$$

where $u = \phi_1(x, y, z)$, $v = \phi_2(x, y, z)$

$$w = \phi_3(x, y, z)$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$



Change of cartesian coordinates (x, y, z) to spherical polar coordinate:

If we have
$$I = \iiint_R f(x, y, z) \, dx \, dy \, dz \quad \text{--- (1)}$$

Then to change above integral in spherical polar coordinates, we have

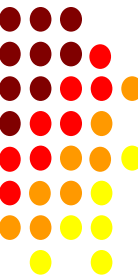
$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$$

$$\text{so } dx \, dy \, dz = r^2 \sin \theta \, dr \, d\theta \, d\phi,$$

so (1) will be

$$I = \iiint_{R'} F(r, \theta, \phi) \, r^2 \sin \theta \, dr \, d\theta \, d\phi \quad \text{--- (2)}$$



Spherical Polar coordinate system:

$$x = r \sin \theta \cos \phi$$

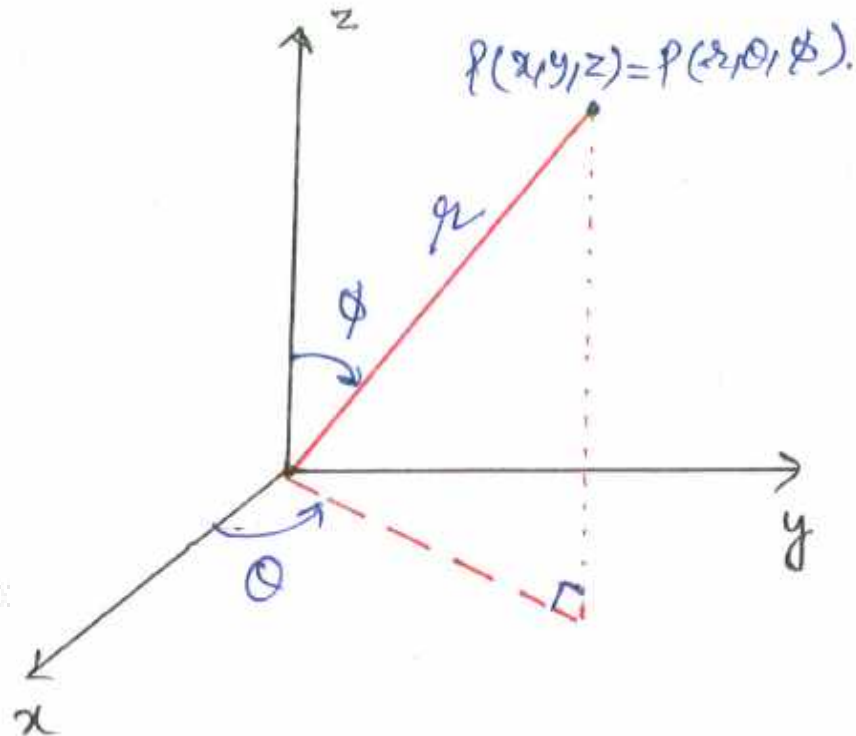
$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

general equation of a sphere is:

$$x^2 + y^2 + z^2 = r^2$$

at origin:



Ques: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz dy dx}{\sqrt{1-x^2-y^2-z^2}}$
by changing to spherical polar coordinates.
[2015-16]

Solⁿ: here region of integration is
bounded by $0 \leq z \leq \sqrt{1-x^2-y^2}$,

$$0 \leq y \leq \sqrt{1-x^2} \text{ \& } 0 \leq x \leq 1$$

so $(x^2+y^2+z^2=1^2)$, sphere of radius 1

region is in 1st octant only

$$\begin{aligned} I &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz dy dx}{\sqrt{1-x^2-y^2-z^2}} \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\rho^2 \sin \theta}{\sqrt{1-\rho^2}} d\rho d\theta d\phi \end{aligned}$$



$$\begin{aligned}
 &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left[\frac{1}{\sqrt{1-z^2}} - \sqrt{1-z^2} \right] \sin \theta \, dz \, d\theta \, d\phi \\
 &= \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \left[\sinh^{-1} z - \left(\frac{z\sqrt{1-z^2}}{2} + \frac{1}{2} \sin^{-1} z \right) \right]_0^1 d\theta \, d\phi \\
 &= \int_0^{\pi/2} \int_0^{\pi/2} \sin \theta \left(\frac{\pi}{2} - \frac{\pi}{4} \right) d\theta \, d\phi \\
 &= \frac{\pi}{4} \int_0^{\pi/2} (-\cos \theta)_0^{\pi/2} d\phi \\
 &= \frac{\pi}{4} \int_0^{\pi/2} d\phi \\
 &= \frac{\pi^2}{8}
 \end{aligned}$$



Example- Evaluate $\iiint \frac{z^2 dx dy dz}{x^2 + y^2 + z^2}$ over the volume of the sphere $x^2 + y^2 + z^2 = 2$.

Solution- Here, we have $I = \iiint \frac{z^2 dx dy dz}{x^2 + y^2 + z^2} \dots(1)$

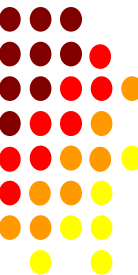
Putting $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, $dx dy dz = r^2 \sin \theta dr d\theta d\phi$ in (1), we get

[The limits r , θ and ϕ over the first octant of $x^2 + y^2 + z^2 = r^2$ are $0, \sqrt{2}$; $0, \frac{\pi}{2}$ and $0, \frac{\pi}{2}$].

$$I = 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \frac{r^4 \cos^2 \theta \sin \theta}{r^2} dr d\theta d\phi$$



$$\begin{aligned}
 I &= 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \frac{r^4 \cos^2 \theta \sin \theta}{r^2} dr d\theta d\phi \\
 &= 8 \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin \theta d\theta \cdot \int_0^{\sqrt{2}} r^2 dr \\
 &= 8 [\phi]_0^{\pi/2} \left[-\frac{\cos^3 \theta}{3} \right]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^{\sqrt{2}} = 8 \frac{\pi}{2} \cdot \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{8\pi\sqrt{2}}{9}. \quad \text{Ans.}
 \end{aligned}$$



Example

Evaluate the integral $\iiint (x^2 + y^2 + z^2) dx dy dz$ taken over the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$.

Solution. Let us convert the given integral into spherical polar co-ordinates. By putting

$$x = r \sin \theta \cos \phi; \quad y = r \sin \theta \sin \phi; \quad z = r \cos \theta$$

$$\begin{aligned} \iiint (x^2 + y^2 + z^2) dx dy dz &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 (r^2 \sin \theta d\theta d\phi dr) \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^1 r^4 dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \left(\frac{r^5}{5} \right)_0^1 = \frac{1}{5} \int_0^{2\pi} d\phi [-\cos \theta]_0^{\pi} = \frac{2}{5} \int_0^{2\pi} d\phi \\ &= \frac{2}{5} (\phi)_0^{2\pi} = \frac{4\pi}{5} \end{aligned}$$

Ans.



Practice questions

Ques: Evaluate by changing the variables
 $\iint (x+y)^2 dx dy$ where R is the region bounded
by the lines $x+y=0$, $x+y=2$, $3x-2y=0$, $3x-2y=3$
[2013-14], [2020-21]

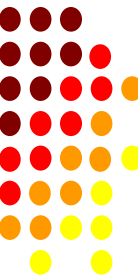
Ques: Evaluate $\iint (x-y)^4 \exp(x+y) dx dy$, where
 R is the square in the $x-y$ plane with vertices
at $(1,0)$, $(2,1)$, $(1,2)$ & $(0,1)$. [2012-13]



Practice question

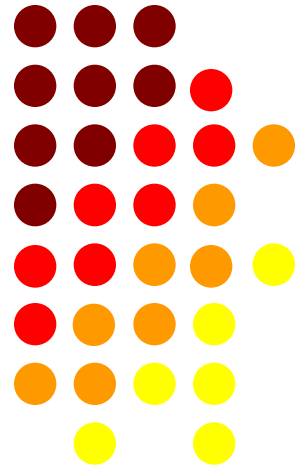
Ques: If the volume of an object expressed in the spherical coordinates as following:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 \sin\phi \, dr \, d\phi \, d\theta . \text{ Evaluate the value of } V. \quad [2016-17]$$



Unit IV
(Multivariate Calculus-I)
L-33

**Gamma Function and
Beta Function**



Gamma Function

If n is positive, then the definite integral $\int_0^{\infty} e^{-x} x^{n-1} dx$, which is a function of n , is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, n > 0.$$

$$\text{In particular, } \Gamma(1) = \int_0^{\infty} e^{-x} dx = \left| -e^{-x} \right|_0^{\infty} = 1.$$

Some Basic Formulae

1. $\Gamma(n + 1) = n\Gamma(n)$

2. $\Gamma(n + 1) = n !$ when n is a positive integer

3. If n is a positive fraction, then by repeated application of above formula, we get

$$\Gamma(n) = (n - 1)(n - 2) \times \text{go on decreasing by 1} \dots\dots$$

the series of factors being continued so long as the factors remain positive, multiplied by Γ (last factor),

$$\Gamma\left(\frac{11}{4}\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4} \cdot \frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

4. $\left|\frac{1}{2}\right| = \sqrt{\pi}$



TRANSFORMATION OF GAMMA FUNCTION

We know that $\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx$... (1)

(i) Replace x by $k y$, so that $dx = k dy$; then (1) becomes

$$\int_0^{\infty} e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$$

(ii) Putting $e^{-x} = y$, so that $-e^{-x} dx = dy$ and $-x = \log y$, $x = \log \frac{1}{y}$, (1) becomes

$$\begin{aligned}\Gamma n &= - \int_1^0 \left(\log \frac{1}{y} \right)^{n-1} y \cdot \frac{dy}{e^{-x}} \\ &= \int_0^1 \left(\log \frac{1}{y} \right)^{n-1} y \cdot \frac{dy}{y} \\ &= \int_0^1 \left(\log \frac{1}{y} \right)^{n-1} dy\end{aligned}$$



Example , Evaluate $\sqrt{-\frac{1}{2}}$.

Solution. $\sqrt{n+1} = n\sqrt{n}$

$$\sqrt{-\frac{1}{2}+1} = -\frac{1}{2}\sqrt{-\frac{1}{2}} \Rightarrow \sqrt{\frac{1}{2}} = -\frac{1}{2}\sqrt{-\frac{1}{2}} \Rightarrow \sqrt{\pi} = -\frac{1}{2}\sqrt{-\frac{1}{2}} \Rightarrow \sqrt{-\frac{1}{2}} = -2\sqrt{\pi} \text{ Ans.}$$



Beta Function

If m, n are positive, then the definite integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$, which is a function of m and n , is called the Beta Function (or Eulerian integral of first kind) and is denoted by $\beta(m, n)$. Thus,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, m > 0, n > 0.$$

Note

- Beta function is a symmetric function. i.e. $B(m, n) = B(n, m)$, where $m > 0, n > 0$



Transformations of Beta Function

$$(1) \quad \boxed{\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx}$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ Put $x = \frac{1}{1+y} \quad \therefore dx = -\frac{1}{(1+y)^2} dy$

$$= \int_{\infty}^0 \frac{1}{(1+y)^{m-1}} \left(1 - \frac{1}{1+y}\right)^{n-1} \left\{ \frac{-1}{(1+y)^2} \right\} dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m-1} (1+y)^{n-1} (1+y)^2} dy = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\therefore \beta(n, m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

But $\beta(m, n) = \beta(n, m)$

$$\therefore \beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$



$$(2) \quad \beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Put $x = \sin^2 \theta$

$\therefore dx = 2 \sin \theta \cos \theta d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2} \theta (1 - \sin^2 \theta)^{n-1} \cdot 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta.$$



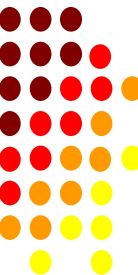
SYMMETRY OF BETA FUNCTION i.e., $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, m > 0, n > 0$$

Since $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\therefore \beta(m, n) = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1} dx = \int_0^1 x^{n-1} (1-x)^{m-1} dx = \beta(n, m)$$

Hence, $\boxed{\beta(m, n) = \beta(n, m).}$



RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad [A.K.T.U. 2018, 2019, 2022]$$

Method-I to Prove Relation

$$\Gamma(n) = k^n \int_0^{\infty} e^{-kx} x^{n-1} dx$$

$$= z^n \int_0^{\infty} e^{-zx} x^{n-1} dx$$

(Replace k by z)

Multiplying both sides by $e^{-z} z^{m-1}$, we get

$$\Gamma(n) \cdot e^{-z} z^{m-1} = \int_0^{\infty} z^n \cdot e^{-zx} \cdot x^{n-1} \cdot e^{-z} \cdot z^{m-1} dx = \int_0^{\infty} z^{n+m-1} e^{-z(1+x)} x^{n-1} dx$$



Integrating both sides w.r.t. z from 0 to ∞ , we get

$$\Gamma(n) \int_0^{\infty} e^{-z} z^{n-1} dz = \int_0^{\infty} x^{n-1} \left\{ \int_0^{\infty} e^{-z(1+x)} z^{m+n-1} dz \right\} dx$$

$$\Rightarrow \Gamma n \Gamma m = \int_0^{\infty} x^{n-1} \left\{ \int_0^{\infty} e^{-y} \cdot \frac{y^{m+n-1}}{(1+x)^{m+n-1}} \frac{dy}{(1+x)} \right\} dx$$

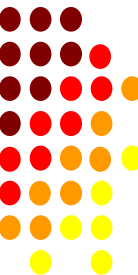
where $z(1+x) = y$ so that $dz = \frac{dy}{1+x}$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} \left\{ \int_0^{\infty} e^{-y} y^{m+n-1} dy \right\} dx$$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx \Gamma(m+n) = \Gamma(m+n) \beta(m, n)$$

$\therefore$

$$\boxed{\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$



Method-II to Prove Relation

We know that $\Gamma(m) = \int_0^{\infty} e^{-t} t^{m-1} dt$

Putting $t = x^2$ so that $dt = 2x dx$

$$\Gamma(m) = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \quad \dots(1)$$

Similarly, $\Gamma(n) = 2 \int_0^{\infty} e^{-y^2} y^{2n-1} dy$

$$\Gamma(m) \Gamma(n) = 4 \int_0^{\infty} e^{-x^2} x^{2m-1} dx \cdot \int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx dy$$



Changing to polar co-ordinates, we have

$$\begin{aligned}
 \Gamma(m) \Gamma(n) &= 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1} \theta \sin^{2n-1} \theta \, dr \, d\theta \\
 &= 4 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr \cdot \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta \, d\theta \quad \dots(2) \\
 &= \left[2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr \right] \cdot \left[2 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta \, d\theta \right] \\
 &= \Gamma(m+n) \beta(m, n) \quad \text{[Using (2) of 4.1\$}
 \end{aligned}$$

Hence,

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$



Some Important Results

$$(i) \quad \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$(ii) \quad \text{If } \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1,$$

then

$$\Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$



Example Prove that $\left| \left(\frac{1}{4} \right) \right| \left| \left(\frac{3}{4} \right) \right| = \pi \sqrt{2}$

Solution. Putting $n = \frac{1}{4}$ in result of example 22, we obtain

$$\left| \left(\frac{1}{4} \right) \right| \left| \left(1 - \frac{1}{4} \right) \right| = \frac{\pi}{\sin \frac{\pi}{4}}$$

$$\Rightarrow \left| \left(\frac{1}{4} \right) \right| \left| \left(\frac{3}{4} \right) \right| = \frac{\pi}{\left(\frac{1}{\sqrt{2}} \right)}$$

$$\Rightarrow \left| \left(\frac{1}{4} \right) \right| \left| \left(\frac{3}{4} \right) \right| = \pi \sqrt{2}$$

Proved.



DUPLICATION FORMULA

$\Gamma(m) \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{(2)^{2m-1}} \Gamma(2m)$ where m is positive. (M.T.U. 2013;

Proof. We have

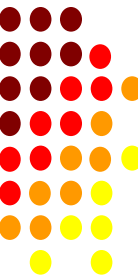
$$\int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{\Gamma(m) \Gamma(n)}{2 \Gamma(m+n)} \quad (1)$$

Putting $2n - 1 = 0$ or $n = \frac{1}{2}$ in (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1} \theta d\theta = \frac{\Gamma(m) \sqrt{\pi}}{2 \Gamma\left(m + \frac{1}{2}\right)} \quad (2)$$

Again putting $n = m$ in equation (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta = \frac{(\Gamma m)^2}{2 \Gamma(2m)}$$



$$\frac{1}{2^{2m-1}} \int_0^{\pi/2} (2 \sin \theta \cos \theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2 \Gamma(2m)}$$

$$\frac{1}{2^{2m}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} \cdot 2d\theta = \frac{(\Gamma m)^2}{2 \Gamma(2m)}$$

Putting $2\theta = \phi$ so that $2 d\theta = d\phi$, this reduces to

$$\frac{1}{2^{2m}} \int_0^{\pi} \sin^{2m-1} \phi d\phi = \frac{(\Gamma m)^2}{2 \Gamma(2m)}$$

$$\frac{2}{2^{2m}} \int_0^{\pi/2} \sin^{2m-1} \phi d\phi = \frac{(\Gamma m)^2}{2 \Gamma(2m)}$$

Replacing ϕ by θ , we finally obtain

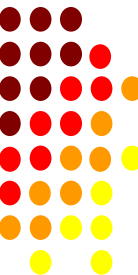
$$\int_0^{\pi/2} \sin^{2m-1} \theta d\theta = \frac{2^{2m-1} (\Gamma m)^2}{2 \Gamma(2m)} \quad (3)$$

From (2) and (3), we get

$$\frac{\Gamma(m) \sqrt{\pi}}{2 \Gamma\left(m + \frac{1}{2}\right)} = \frac{2^{2m-1} (\Gamma m)^2}{2 \Gamma(2m)}$$

$\Rightarrow$

$$\Gamma(m) \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$



Home Work

Prove the following problems:

$$1 \quad \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})} = (2)^{1/3} \sqrt{\pi} \quad (A.K.T.U. 2017)$$

$$2 \quad \Gamma(-\frac{3}{2}) = \frac{4}{3} \sqrt{\pi}$$

$$3 \quad \frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p} = \frac{B(p, q)}{p+q}, \quad (p > 0, q > 0)$$

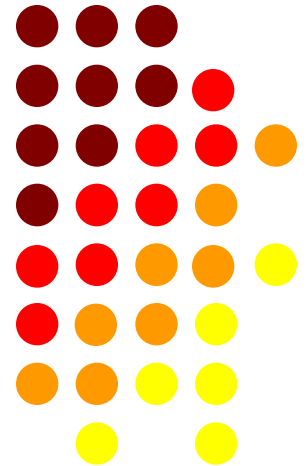
$$4 \quad \beta(m, n) = \beta(m+1, n) + \beta(m, n+1), \text{ for } m > 0, n > 0$$



Unit IV
(Multivariate Calculus-I)
L-34



Problems Based Upon
Beta & Gamma
Functions



Gamma Function

If n is positive, then the definite integral $\int_0^{\infty} e^{-x} x^{n-1} dx$, which is a function of n , is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, n > 0.$$

$$\text{In particular, } \Gamma(1) = \int_0^{\infty} e^{-x} dx = \left| -e^{-x} \right|_0^{\infty} = 1.$$

Beta Function

If m, n are positive, then the definite integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$, which is a function of m and n , is called the Beta Function (or Eulerian integral of first kind) and is denoted by $\beta(m, n)$. Thus,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, m > 0, n > 0.$$

Note

- Beta function is a symmetric function. i.e. $B(m, n) = B(n, m)$, where $m > 0, n > 0$



Using Beta and Gamma functions, evaluate:

$$(i) \int_0^{\infty} x^{1/4} e^{-\sqrt{x}} dx$$

$$(ii) \int_0^1 \left(\frac{x^3}{1-x^3} \right)^{1/2} dx \quad (\text{A.K.T.U. 2014, 2018})$$

$$(iii) \int_0^1 x^5 (1-x^3)^{10} dx$$



(i) Let

$$I = \int_0^{\infty} x^{1/4} e^{-\sqrt{x}} dx$$

Put $\sqrt{x} = y \Rightarrow x = y^2$ so that $dx = 2y dy$ then equation (1) becomes

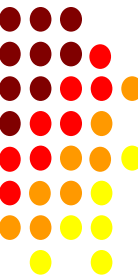
$$I = \int_0^{\infty} y^{1/2} e^{-y} \cdot 2y dy = 2 \int_0^{\infty} e^{-y} y^{3/2} dy$$

$$= 2 \int_0^{\infty} e^{-y} y^{(5/2)-1} dy = 2 \Gamma(5/2)$$

| By definition

$$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} = \frac{3}{2} \sqrt{\pi}$$

| $\because \Gamma(n+1) = n \Gamma(n)$



(ii) Let

$$I = \int_0^1 x^{3/2} (1-x^3)^{-1/2} dx$$

...(1) 

Put $x^3 = y \Rightarrow x = y^{1/3}$ so that $dx = \frac{1}{3} y^{-2/3} dy$ then equation (1) becomes

$$I = \int_0^1 y^{1/2} (1-y)^{-1/2} \cdot \frac{1}{3} y^{-2/3} dy = \frac{1}{3} \int_0^1 y^{-1/6} (1-y)^{-1/2} dy$$

$$= \frac{1}{3} \int_0^1 y^{\left(\frac{5}{6}\right)-1} (1-y)^{\left(\frac{1}{2}\right)-1} dy = \frac{1}{3} \beta\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$= \frac{1}{3} \frac{\Gamma(5/6) \Gamma(1/2)}{\Gamma(4/3)}$$

$$\left| \because \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \right.$$



$$= \frac{\sqrt{\pi}}{3} \cdot \frac{\Gamma(5/6)}{\frac{1}{3}\Gamma(1/3)} = \sqrt{\pi} \cdot \frac{\Gamma(5/6)\Gamma(1/6)\Gamma(2/3)}{\Gamma(1/6)\Gamma(1/3)\Gamma(2/3)}$$

$$= \sqrt{\pi} \cdot \frac{\Gamma(2/3)}{\Gamma(1/6)} \cdot \frac{\pi}{\sin \frac{\pi}{6}} \cdot \frac{\sin \frac{\pi}{3}}{\pi}$$

$$= \sqrt{3\pi} \frac{\Gamma(2/3)}{\Gamma(1/6)}$$

(iii) Solution is similar as (ii)



Evaluate:

$$\int_0^{\infty} \frac{x^8 (1 - x^6)}{(1 + x)^{24}} dx \quad (\text{M.T.U. 2013})$$

Sol. (i)
$$I = \int_0^{\infty} \frac{x^8}{(1 + x)^{24}} dx - \int_0^{\infty} \frac{x^{14}}{(1 + x)^{24}} dx$$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1 + x)^{9+15}} dx - \int_0^{\infty} \frac{x^{15-1}}{(1 + x)^{15+9}} dx$$

$$= \beta(9, 15) - \beta(15, 9) = 0$$



Evaluate: $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$ (G.B.T.U. 2011)

Sol. $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^2 = \tan \theta \Rightarrow x = \sqrt{\tan \theta}$

$$\therefore dx = \frac{1}{2\sqrt{\tan \theta}} \sec^2 \theta d\theta$$

$$I = \int_0^{\pi/4} \frac{1}{\sec \theta} \cdot \frac{\sec^2 \theta}{2\sqrt{\tan \theta}} d\theta$$



$$= \frac{1}{2} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin \theta \cos \theta}} = \frac{1}{2} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Put $2\theta = t \quad \therefore \quad d\theta = \frac{dt}{2}$

$$I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \, dt$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{(-1/2)+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{(-1/2)+0+2}{2}\right)} = \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma(1/4) \Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{\Gamma(1/4)^2}{\Gamma(1/4) \Gamma(3/4)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \cdot \frac{\Gamma(1/4)^2}{\left(\frac{\pi}{\sin \pi/4}\right)}$$

$$= \frac{1}{8\sqrt{\pi}} \left(\Gamma \frac{1}{4}\right)^2$$



Example

Prove the following $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} \times \int_0^{\pi/2} \sqrt{\sin \theta} d\theta = \pi$.

(U.P.T.U. 2014)

Proof: We have $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} \times \int_0^{\pi/2} \sqrt{\sin \theta} d\theta$

$$= \int_0^{\pi/2} \sin^{-1/2} \theta \cos^0 \theta d\theta \times \int_0^{\pi/2} \sin^{1/2} \theta \cos^0 \theta d\theta$$

$$= \frac{\Gamma\left(-\frac{1}{2} + 1\right) \Gamma\left(\frac{0 + 1}{2}\right)}{2 \Gamma\left(-\frac{1}{2} + 0 + 2\right)} \times \frac{\Gamma\left(\frac{1}{2} + 1\right) \Gamma\left(\frac{0 + 1}{2}\right)}{2 \Gamma\left(\frac{1}{2} + 0 + 2\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{3}{4}\right)} \times \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right) \sqrt{\pi}}{4} \times \frac{\sqrt{\pi}}{\frac{1}{4} \Gamma\left(\frac{1}{4}\right)} = \pi$$



Example

Prove that.

$$\beta(l, m) \cdot \beta(l + m, n) \cdot \beta(l + m + n, p) = \frac{\Gamma l \Gamma m \Gamma n \Gamma p}{\Gamma(l + m + n + p)}$$

Proof:

$$\begin{aligned} \text{LHS} &= \beta(l, m) \cdot \beta(l + m, n) \cdot \beta(l + m + n, p) \\ &= \frac{\Gamma l \Gamma m}{\Gamma(l + m)} \cdot \frac{\Gamma(l + m) \Gamma n}{\Gamma(l + m + n)} \cdot \frac{\Gamma(l + m + n) \Gamma p}{\Gamma(l + m + n + p)} \\ &= \frac{\Gamma l \Gamma m \Gamma n \Gamma p}{\Gamma(l + m + n + p)} = \text{RHS} \end{aligned}$$



Home Work

Show the following:

$$1 \quad \int_0^{\pi/2} \sqrt{\tan \theta} \, d\theta = \int_0^{\pi/2} \sqrt{\cot \theta} \, d\theta = \frac{\pi}{\sqrt{2}}$$

$$2 \quad \int_0^{\infty} \frac{x^4 (1+x^5)}{(1+x)^{15}} dx = \frac{1}{5005}$$

$$3 \quad \int_0^2 (8-x^3)^{-1/3} dx = \frac{2\pi}{3\sqrt{3}}$$

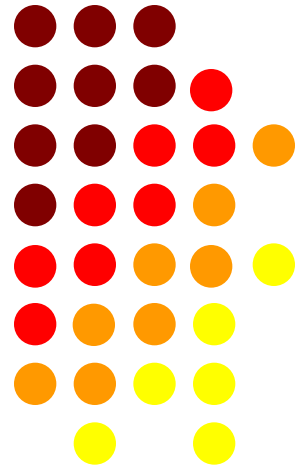
$$4 \quad \int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$$

$$5 \quad \beta(m,n) = \beta(m+1,n) + \beta(m,n+1), \text{ for } m > 0, n > 0$$



Unit IV
(Multivariate Calculus-I)
L-35

**DIRICHLET'S
INTEGRAL**



DIRICHLET'S INTEGRAL

If V is a region bounded by $x \geq 0$, $y \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\overline{(l)} \overline{(m)} \overline{(n)}}{\overline{(l+m+n+1)}}$$

This integral is known as **Dirichlet's integral** This is an important integral useful in evaluating multiple integrals.



Example 1. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol. The volume in the positive octant will be

$$V = \iiint dx dy dz$$

For points within positive octant, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

Put $\frac{x^2}{a^2} = u$ or $x = a\sqrt{u}$, $y = b\sqrt{v}$, $z = c\sqrt{w}$,

$\therefore dx = \frac{a}{2}u^{-\frac{1}{2}} du$, $dy = \frac{b}{2}v^{-\frac{1}{2}} dv$, $dz = \frac{c}{2}w^{-\frac{1}{2}} dw$

$$V = \frac{abc}{8} \iiint u^{\left(\frac{1}{2}-1\right)} v^{\left(\frac{1}{2}-1\right)} w^{\left(\frac{1}{2}-1\right)} du dv dw, \text{ where } u + v + w \leq 1$$

$$= \frac{abc}{8} \cdot \frac{\left[\left(\frac{1}{2}\right)\right] \cdot \left[\left(\frac{1}{2}\right)\right] \cdot \left[\left(\frac{1}{2}\right)\right]}{\left[1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right]}$$

$$= \frac{abc}{8} \frac{(\sqrt{\pi})^3}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} = \frac{\pi abc}{6}$$

$$\therefore \text{Total volume} = 8 \times \frac{\pi abc}{6} = \frac{4}{3} \pi abc.$$



Example 2. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axis in A, B and C . Apply Ditchlet's integral to find the volume of the tetrahedron $OABC$. Also find its mass if the density at any point is $kxyz$.

Sol. Let $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$, then $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w \leq 1$

Also, $dx = a du, dy = b dv, dz = c dw$.

$$\begin{aligned} \text{Volume } OABC &= \iiint_D dx dy dz \\ &= \iiint_D abc du dv dw, \text{ where } u + v + w < 1 \\ &= abc \iiint_D u^{1-1} v^{1-1} w^{1-1} du dv dw \\ &= abc \frac{\overline{(1)} \overline{(1)} \overline{(1)}}{\overline{(1+1+1+1)}} = \frac{abc}{3!} = \frac{abc}{6} \end{aligned}$$

$$\begin{aligned} \text{Mass} &= \iiint_D kxyz dx dy dz = \iiint_{D'} k (au)(bv)(cw) abc du dv dw \\ &= ka^2 b^2 c^2 \iiint_{D'} u^{2-1} v^{2-1} w^{2-1} du dv dw \\ &= ka^2 b^2 c^2 \frac{\overline{2} \overline{2} \overline{2}}{\overline{(2+2+2+1)}} = ka^2 b^2 c^2 \frac{1!1!1!}{6!} = \frac{ka^2 b^2 c^2}{720}. \end{aligned}$$

Example 3. Find the volume of the solid surrounded by the surface

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1.$$

(U.P.T.U. 2008)

Sol. Let

$$\begin{aligned} \left(\frac{x}{a}\right)^{2/3} = u &\Rightarrow x = au^{3/2} &\therefore dx = \frac{3a}{2} u^{1/2} du \\ \left(\frac{y}{b}\right)^{2/3} = v &\Rightarrow y = bv^{3/2} &\therefore dy = \frac{3b}{2} v^{1/2} dv \\ \left(\frac{z}{c}\right)^{2/3} = w &\Rightarrow z = cw^{3/2} &\therefore dz = \frac{3c}{2} w^{1/2} dw \end{aligned}$$

For the positive octant,

$$\begin{aligned} x \geq 0 &\Rightarrow au^{3/2} \geq 0 &\Rightarrow u \geq 0 \\ y \geq 0 &\Rightarrow bv^{3/2} \geq 0 &\Rightarrow v \geq 0 \\ z \geq 0 &\Rightarrow cw^{3/2} \geq 0 &\Rightarrow w \geq 0 \end{aligned}$$

Then, we have $u + v + w = 1$, $u \geq 0$, $v \geq 0$, $w \geq 0$.

Required volume

$$\begin{aligned} &= 8 \iiint dx dy dz \\ &= 8 \iiint \frac{3a}{2} u^{1/2} \cdot \frac{3b}{2} v^{1/2} \cdot \frac{3c}{2} w^{1/2} du dv dw \\ &= 27 abc \iiint u^{\frac{3}{2}-1} v^{\frac{3}{2}-1} w^{\frac{3}{2}-1} du dv dw \\ &= 27 abc \cdot \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} = \frac{4\pi abc}{35} \end{aligned}$$



Example 3 . Find the mass of an octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the density at any point being $\rho = kxyz$.

Solution.

$$\begin{aligned} \text{Mass} &= \iiint \rho \, dv = \iiint (kxyz) \, dx \, dy \, dz \\ &= k \iiint (x \, dx) (y \, dy) (z \, dz) \end{aligned} \quad \dots (1)$$

Putting $\frac{x^2}{a^2} = u, \frac{y^2}{b^2} = v, \frac{z^2}{c^2} = w$ and $u + v + w = 1$

so that $\frac{2x \, dx}{a^2} = du, \frac{2y \, dy}{b^2} = dv, \frac{2z \, dz}{c^2} = dw$

$$\begin{aligned} \text{Mass} &= k \iiint \left(\frac{a^2 \, du}{2} \right) \left(\frac{b^2 \, dv}{2} \right) \left(\frac{c^2 \, dw}{2} \right) \\ &= \frac{k a^2 b^2 c^2}{8} \iiint du \, dv \, dw \quad \text{where } u + v + w \leq 1 \\ &= \frac{k a^2 b^2 c^2}{8} \iiint u^{1-1} v^{1-1} w^{1-1} \, du \, dv \, dw \\ &= \frac{k a^2 b^2 c^2}{8} \frac{[1] [1] [1]}{[3+1]} = \frac{k a^2 b^2 c^2}{8 \times 6} \\ &= \frac{k a^2 b^2 c^2}{48} \end{aligned}$$

Ans.



Example4 The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is $kxyz$.
(A.K.T.U. 2012, 2018)

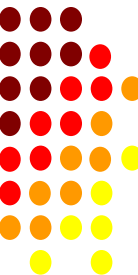
Sol. Put $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$, then $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w \leq 1$.

Also, $dx = a du, dy = b dv, dz = c dw$.

$$\begin{aligned}
 \text{Volume OABC} &= \iiint_D dx dy dz \\
 &= \iiint_{D'} abc du dv dw, \quad \text{where } u + v + w \leq 1 \\
 &= abc \iiint_{D'} u^{1-1} v^{1-1} w^{1-1} du dv dw \\
 &= abc \frac{\Gamma(1) \Gamma(1) \Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3!} = \frac{abc}{6}
 \end{aligned}$$

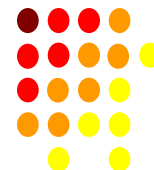


$$\begin{aligned}
 \text{Mass} &= \iiint_D kxyz \, dx \, dy \, dz = \iiint_{D'} k(au)(bv)(cw) abc \, du \, dv \, dw \\
 &= ka^2b^2c^2 \iiint_{D'} u^{2-1}v^{2-1}w^{2-1} \, du \, dv \, dw \\
 &= ka^2b^2c^2 \frac{\Gamma(2)\Gamma(2)\Gamma(2)}{\Gamma(2+2+2+1)} = ka^2b^2c^2 \frac{1!1!1!}{6!} = \frac{ka^2b^2c^2}{720}.
 \end{aligned}$$



Home Work

1. Evaluate $\iiint xyz \, dx \, dy \, dz$ for all positive value of variables of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. Evaluate $\iiint_V (ax^2 + by^2 + cz^2) \, dx \, dy \, dz$ where V is the region bounded by $x^2 + y^2 + z^2 \leq 1$.
3. Compute $\iiint_V x^2 \, dx \, dy \, dz$ over volume of tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
4. Evaluate $\iiint_V x^2 yz \, dx \, dy \, dz$ throughout the volume bounded by planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.



ANSWERS

$$1 \quad \frac{a^2 b^2 c^2}{48}$$

$$2 \quad \frac{\pi(a + b + c)}{30}$$

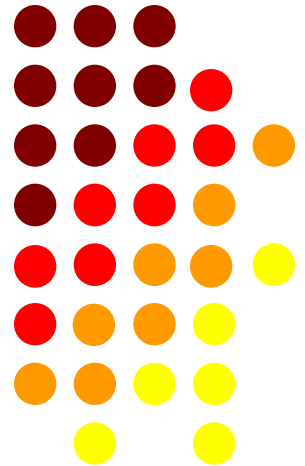
$$3 \quad \frac{a^3 b c}{60}$$

$$4 \quad \frac{a^3 b^2 c^2}{2520}$$



Unit IV
(Multivariate Calculus-I)
L-36

**LIOUVILLE'S EXTENSION
OF DIRICHLET'S
INTEGRAL**



DIRICHLET'S INTEGRAL

If V is a region bounded by $x \geq 0$, $y \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\overline{(l)} \overline{(m)} \overline{(n)}}{\overline{(l+m+n+1)}}$$

This integral is known as **Dirichlet's integral** This is an important integral useful in evaluating multiple integrals.



LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables x, y, z are all positive such that $h_1 < (x + y + z) < h_2$ then

$$\iiint f(x + y + z) x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\Gamma(l) \Gamma(m) \Gamma(n)}{\Gamma(l + m + n)} \int_{h_1}^{h_2} f(u) u^{l+m+n-1} du.$$



Example 1. Evaluate $\iiint \log(x+y+z) dx dy dz$, the integral extending over all positive and zero values of x, y, z subject to $x+y+z < 1$.

Sol. $0 \leq x+y+z < 1$

$$\begin{aligned} \therefore \iiint \log(x+y+z) dx dy dz &= \iiint x^{1-1} y^{1-1} z^{1-1} \log(x+y+z) dx dy dz \\ &= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(3)} \int_0^1 t^{1+1+1-1} \log t dt \quad | \text{ By Liouville's extension} \end{aligned}$$

$$= \frac{1}{2} \int_0^1 t^2 \log t dt = \frac{1}{2} \left[\left(\frac{t^3}{3} \log t \right)_0^1 - \int_0^1 \frac{t^3}{3} \cdot \frac{1}{t} dt \right] = \frac{1}{2} \left[-\frac{1}{3} \left(\frac{t^3}{3} \right)_0^1 \right] = -\frac{1}{18}.$$



Example . Show that $\iiint \frac{dx dy dz}{(x+y+z+1)^3} = \frac{1}{2} \log 2 - \frac{5}{16}$, the integral being taken

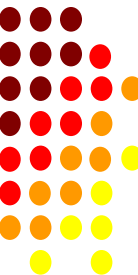
throughout the volume bounded by planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Sol. $0 \leq x+y+z \leq 1$

$$\begin{aligned} \therefore \iiint \frac{dx dy dz}{(x+y+z+1)^3} &= \iiint \frac{x^{1-1} y^{1-1} z^{1-1}}{(x+y+z+1)^3} dx dy dz \\ &= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3} u^{1+1+1-1} du \\ &= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^3} du \quad | \text{ By Liouville's extension} \end{aligned}$$

Put $u+1=t$ so that $du=dt$

$$\begin{aligned} \therefore \text{ Required integral } &= \frac{1}{2} \int_1^2 \frac{(t-1)^2}{t^3} dt = \frac{1}{2} \int_1^2 \left(\frac{t^2 - 2t + 1}{t^3} \right) dt \\ &= \frac{1}{2} \int_1^2 \left(\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3} \right) dt = \frac{1}{2} \left[\log t + \frac{2}{t} - \frac{1}{2t^2} \right]_1^2 \end{aligned}$$



Example Find the value of $\iiint \log (x+y+z) dx dy dz$ the integral extending over all positive and zero values of x, y, z subject to the condition $x+y+z < 1$.

Solution. By Liouville's theorem when $0 < x+y+z < 1$

$$\iiint \log (x+y+z) dx dy dz$$

$$= \iiint \log (x+y+z) x^{1-1} y^{1-1} z^{1-1} dx dy dz = \frac{\sqrt[1]{1} \sqrt[1]{1} \sqrt[1]{1}}{\sqrt[1]{1+1+1}} \int_0^1 (\log u) u^{1+1+1-1} du$$

$$= \frac{1}{\sqrt[3]{3}} \int_0^1 u^2 \log u du = \frac{1}{2} \left[\log u \left(\frac{u^3}{3} \right) - \frac{1}{3} \frac{u^3}{3} \right]_0^1 = \frac{1}{2} \left(-\frac{1}{9} \right) = -\frac{1}{18}$$

Ans.



Home work

1. Find the mass of a solid $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$, the density at any point being $\rho = kx^{l-1}y^{m-1}z^{n-1}$ where x, y, z are all positive.
2. Find the mass of the region bounded by ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if the density varies as the square of the distance from the centre. [**Hint.** $\rho = k(x^2 + y^2 + z^2)$]
3. Prove that $\iiint \frac{dx \, dy \, dz}{\sqrt{1 - x^2 - y^2 - z^2}} = \frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real.
4. Evaluate $\iiint \sqrt{\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}} \, dx \, dy \, dz$, integral being taken over all positive values of x, y, z such that $x^2 + y^2 + z^2 \leq 1$.

